CS228 Logic for Computer Science 2020

Lecture 8: k-SAT and XOR SAT

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Topic 8.1

k-sat

k-sat

Definition 8.1

A k-sat formula is a CNF formula and has at most k literals in each of its clauses

Example 8.1

- $(p \wedge q \wedge \neg r)$ *is 1-SAT*
- $(p \vee \neg p) \wedge (p \vee q)$ *is 2-SAT*
- $(p \vee \neg q \vee \neg s) \wedge (p \vee q) \wedge \neg r$ *is 3-SAT*

3-SAT satisfiablity

Theorem 8.1

For each k-SAT formula F there is a 3-SAT formula F' with linear blow up such that F and F' are equivsatisfiable.

Proof.

Consider F a k-SAT formula with $k \geq 4$.
Consider a clause $G = (\ell_1 \vee \cdots \vee \ell_k)$ in F, where ℓ_i are literals.

Let $x_2, \ldots, x_{k-2}$ be variables that do not appear in F.
Let G' be the following set of clauses

$$(\ell_1 \vee \ell_2 \vee x_2) \wedge \bigwedge_{i \in 2..k-3} (\neg x_i \vee x_{i+1} \vee \ell_{i+1}) \wedge (\neg x_{k-2} \vee \ell_{k-1} \vee \ell_k).$$

We show G is sat iff G' is sat. ...

Exercise 8.1

Convert the following CNF in 3-SAT

- $(p \vee \neg q \vee s \vee \neg t) \wedge (\neg q \vee x \vee \neg y \vee z)$

3-SAT satisfiability(cont. I)

Proof(contd. from last slide).

Recall

$$G' = (\ell_1 \vee \ell_2 \vee x_2) \wedge \bigwedge_{i \in 2..k-3} (\neg x_i \vee x_{i+1} \vee \ell_{i+1}) \wedge (\neg x_{k-2} \vee \ell_{k-1} \vee \ell_k).$$

Assume $m \models G'$:

Assume for each $i \in 1..k$, $m(\ell_i) = 0$.
Due to the first clause $m(x_2) = 1$.
Due to ith clause, if $m(x_i) = 1$ then $m(x_{i+1}) = 1$.
Due to induction, $m(x_{k-2}) = 1$.
Due to the last clause of G', $m(x_{k-2}) = 0$. Contradiction.
Therefore, exists $i \in 1..k$ $m(\ell_i) = 1$. Therefore $m \models G$.

...

3-SAT satisfiability(cont. II)

Proof(contd. from last slide).

Recall

$$G' = (\ell_1 \vee \ell_2 \vee x_2) \wedge \bigwedge_{i \in 2..k-2} (\neg x_i \vee x_{i+1} \vee \ell_{i+1}) \wedge (\neg x_{k-2} \vee \ell_{k-1} \vee \ell_k).$$

Assume $m \models G$:

There is a $m(\ell_i) = 1$.

Let $m' \triangleq m[x_2 \mapsto 1, .., x_{i-1} \mapsto 1, x_i \mapsto 0, \dots, x_{k-2} \mapsto 0]$.

Therefore, $m' \models G'_{\text{(why?)}}$.

G' contains $3(k-2)$ literals.

In the worst case, the formula size will increase 3 times. □

Exercise 8.2

a. Complete the above argument.

b. Show a 3-SAT cannot be converted into a 2-SAT via Tseitin's encoding.

c. When is the worst case?

Special classes of formulas

We will discuss the following subclasses whose SAT problems are polynomial

- 2-SAT
- XOR-SAT
- Horn clauses

Topic 8.2

2-SAT

2-SAT

Definition 8.2

A 2-sat formula is a CNF formula that has only binary clauses

We assume that unit clauses are replaced by clauses with repeated literals.

Example 8.2

- $(\neg p \vee q) \wedge (\neg q \vee r) \wedge (\neg r \vee p) \wedge (r \vee q)$ *is a 2-SAT formula*
- $(p \vee p) \wedge (\neg p \vee \neg p)$ *is a 2-SAT formula*

Implication graph

Definition 8.3

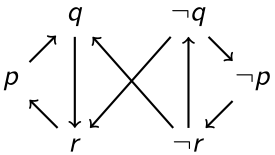
Let F be a 2-SAT formula s.t. $\mathbf{Vars}(F) = \{p_1, \dots, p_n\}$.
The implication graph (V, E) *for F is defined as follows.*

- $V = \{p_1, \dots, p_n, \neg p_1, \dots, \neg p_n\}$
- $E = \{(\bar{\ell_1}, \ell_2), (\bar{\ell_2}, \ell_1) | (\ell_1 \vee \ell_2) \in F\}$,

where $\bar{p} = \neg p$ *and* $\overline{\neg p} = p$.

Example 8.3

Consider $(\neg p \vee q) \wedge (\neg q \vee r) \wedge (\neg r \vee p) \wedge (r \vee q)$.



Exercise 8.3

Draw implication graphs of the following

1. $(p \vee q) \wedge (\neg p \vee \neg q)$
2. $(p \vee \neg q) \wedge (q \vee p) \wedge (\neg p \vee \neg r) \wedge (r \vee \neg p)$
3. $(p \vee p) \wedge (\neg p \vee \neg p)$
4. $(p \vee \neg p) \wedge (p \vee \neg p)$

Properties of implication graph

Consider a formula F and its implication graph (V, E).

Theorem 8.2

If there is a path from ℓ_1 to ℓ_2 in (V, E) then there is a path from $\bar{\ell_2}$ to $\overline{\ell_1}$.

Exercise 8.4

a. Prove the above theorem.
b. Does the above theorem imply
if there is a path from p to $\neg p$ in (V, E) then there is a path from $\neg p$ to p?

Theorem 8.3

For every strongly connected component(scc) $S \subseteq V$ in (V, E), there is another scc S^c, called complementary component, that has exactly the set of literals that are negation of the literals in S.

Proof.

Due to theorem 8.2. □

Properties of implication graph (contd.)

Theorem 8.4
For each model $m \models F$, if there is a path from ℓ_1 to ℓ_2 in (V, E) then if $m(\ell_1) = 1$ then $m(\ell_2) = 1$.

Theorem 8.5
For each model $m \models F$ and each scc S in (V, E),
either for each $\ell \in S$ $m(\ell) = 1$ or for each $\ell \in S$ $m(\ell) = 0$.

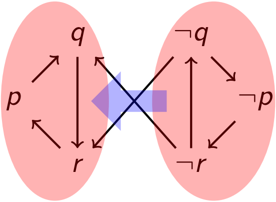
Exercise 8.5
Prove the above theorems.

Reduced implication graph

Definition 8.4
For an implication graph (V, E), the reduced implication DAG (V^R, E^R) is defined as follows.

- $V^R = \{S | S \text{ is a scc in (V,E)}\}$
- $E^R = \{(S, S') | \text{there are } \ell \in S \text{ and } \ell' \in S' \text{ s.t. } (\ell, \ell') \in E\}$

Example 8.4



Theorem 8.6
If $(S, S') \in E^R$ then $(S'^c, S^c) \in E^R$.

Exercise 8.6 *Prove the above theorem.*

Commentary: (V^R, E^R) is a graph over scc's of (V, E). Please notice that (V^R, E^R) will always be a directed acyclic graph (DAG).

2-SAT satisfiablity

Theorem 8.7

A 2-SAT formula F is unsat iff there is a scc S in its implication graph (V, E) such that $\{p, \neg p\} \subseteq S$ for some p.

Proof.

Reverse direction

There must be a path that goes from p to $\neg p$.

$$\begin{tikzcd}
& \ell_1 \arrow[rr, dashed] & & \ell_n \arrow[dr] & \\
p \arrow[ur] & & & & \neg p \arrow[dl] \\
& \ell'_1 \arrow[ul] & & \ell'_n \arrow[ll, dashed] &
\end{tikzcd}$$

Therefore, if p is true then $\neg p$ is true. Therefore, p must be false.
Similarly, if p is false then $\neg p$ is false. Therefore, p must is true.
F is unsat. ...

2-SAT satisfiablity(contd.)

Proof(contd.)

Fwd direction: Let us assume there is no such S.

We will construct a model of F as follows.

1. Initially all literals are unassigned.
2. While(some scc in V^R is unassigned)
   2.1 Let $S \in V^R$ be an unassigned scc whose all children are assigned 1.
   2.2 Assign literals of S to 1. Consequently, S^c is assigned 0.

...

2-SAT satisfiablity(contd.)

Proof(contd.)

We need to show that step 2.1 always finds S with all children assigned 1.

claim: at step 2.1, there is an unassigned node whose all children are assigned

Choose an unassigned node.

Descend down if there is an unassigned child.

Since the DAG is finite, the process will terminate.

claim: an unassigned node can not have a child that is assigned 0.

If S is assigned 1, all its children are already 1.

Therefore, all the parents of S^c are already assigned 0(due to theorem 8.6).

Therefore, no node with 0 assignment has an unassigned parent. □

Exercise 8.7

Show that the procedure produces a satisfying model.

2-SAT is polynomial

Theorem 8.8

A 2-SAT satisfiability problem can be solved in linear time.

Proof.

Due to the previous theorem, 2-SAT satisfiability problem is polynomial. □

Attendance quiz

In the implication graph of $(\neg x \vee \neg y) \wedge (\neg y \vee \neg z) \wedge (\neg z \vee \neg x)$, which of the following holds?

$\neg x$ is reachable from y
$\neg x$ is reachable from z
$\neg y$ is reachable from z
$\neg y$ is reachable from x
$\neg z$ is reachable from x
$\neg z$ is reachable from y
$\neg z$ is reachable from z
$\neg z$ is reachable from $\neg x$
$\neg z$ is reachable from $\neg y$
$\neg x$ is reachable from x
$\neg x$ is reachable from $\neg y$
$\neg x$ is reachable from $\neg z$
$\neg y$ is reachable from y
$\neg y$ is reachable from $\neg x$
$\neg y$ is reachable from $\neg z$

Exercise: 2-SAT solving

Exercise 8.8

Find a satisfying assignment of the following formula

1. $(\neg x \vee \neg y) \wedge (\neg y \vee \neg z) \wedge (\neg z \vee \neg x) \wedge (x \vee \neg w) \wedge (y \vee \neg w) \wedge (z \vee \neg w)$
2. $(p_0 \vee p_2) \wedge (p_0 \vee \neg p_3) \wedge (p_1 \vee \neg p_3) \wedge (p_1 \vee \neg p_4) \wedge (p_2 \vee \neg p_4) \wedge$
   $(p_0 \vee \neg p_5) \wedge (p_1 \vee \neg p_5) \wedge (p_2 \vee \neg p_5) \wedge (p_3 \vee p_6) \wedge (p_4 \vee p_6) \wedge (p_5 \vee p_6)$

Topic 8.3

XOR SAT

XOR-SAT

Definition 8.5
A formula is XOR-SAT if it is a conjunction of xors of literals.

Example 8.5
$(p \oplus r \oplus s) \wedge (q \oplus \neg r \oplus s) \wedge (p \oplus q \oplus \neg s) \wedge (p \oplus \neg q \oplus \neg r)$
is a XOR-SAT formula.

Solving XOR-SAT

Since xors are negation of equality, we may eliminate variables via substitution.

Theorem 8.9
For a variable, p, xor formula G, and XOR-SAT formula F,
$(p \oplus G) \wedge F$ is sat iff $F[\neg G/p]$ is sat

Exercise 8.9
Prove the above theorem.

Example : solving XOR-SAT

Example 8.6

$(p \oplus r \oplus s) \land (q \oplus \neg r \oplus s) \land (p \oplus q \oplus \neg s) \land (p \oplus \neg q \oplus \neg r)$

Eliminate p:

Due to the first xor: $p \Leftrightarrow \neg r \oplus s$

After substitution: $(q \oplus \neg r \oplus s) \land (\neg r \oplus s \oplus q \oplus \neg s) \land (\neg r \oplus s \oplus \neg q \oplus \neg r)$

Simplification: $(q \oplus \neg r \oplus s) \land (\neg r \oplus \neg q) \land (s \oplus \neg q)$

Eliminate r:

Due to the second xor: $r \Leftrightarrow \neg q$

After substitution: $(q \oplus \neg\neg q \oplus s) \land (s \oplus \neg q)$

Simplification: $s \land (s \oplus \neg q)$

Eliminate q:

Due to the second xor: $q \Leftrightarrow s$

After substitution: s

Solution:

$m(s) = 1$ $\quad\quad m(q) = m(s) = 1$

$m(r) = m(\neg q) = 0$ $\quad\quad m(p) = m(\neg r \oplus s) = 0$

Exercise: XOR-SAT

Exercise 8.10

Find a satisfying assignment of the following formula

- $(p \oplus r \oplus s) \land (q \oplus r \oplus s) \land (\neg p \oplus q \oplus \neg s) \land (p \oplus \neg q \oplus \neg r)$

Topic 8.4

Horn Clauses

Horn clauses

Definition 8.6

A Horn clause is a clause that has the following form

$$\neg p_1 \vee \cdots \vee \neg p_n \vee q,$$

*where $p_1, \ldots, p_n \in$ **Vars**, and $q \in$ **Vars** $\cup \{\bot\}$.*
A Horn formula is a set of Horn clauses, which is interpreted as conjunction of the Horn clauses.
The clauses with $\bot$ literals are called goal clauses and others are called implication clauses.

Example 8.7

The following set is a Horn formula

$\{p, \quad \neg q \vee \neg r \vee \neg t \vee p,$
$\quad \neg p \vee q, \quad \neg p \vee \neg r \vee t,$
$\quad \neg p \vee \neg q \vee t, \quad \neg r \vee \bot,$
$\quad \neg p \vee \neg q \vee \neg t \vee \bot\}$

Implication view of the horn clauses

We may view a Horn clause

$$\neg p_1 \vee \cdots \vee \neg p_n \vee q$$

as

$$p_1 \wedge \cdots \wedge p_n \Rightarrow q.$$

Example 8.8

The following is an implication view of a Horn formula

$$\begin{aligned}\{&\top \Rightarrow p, \quad q \wedge r \wedge t \Rightarrow p,\\ &p \Rightarrow q, \quad p \wedge r \Rightarrow t,\\ &\neg p \wedge q \Rightarrow t, \quad r \Rightarrow \bot,\\ &p \wedge q \wedge t \Rightarrow \bot\}\end{aligned}$$

Note $\top \Rightarrow p$ means p, which is a Horn clause without negative literals

Horn satisfiability

Algorithm 8.1: $\textsc{HornSAT}$(Hs,Gs)

Input: Hs: implication clauses, Gs : goal clauses

Output: model/unsat

$m := \lambda x.0$;

while $m \not\models (p_1 \land .. \land p_n \Rightarrow p) \in Hs$ **do**

  $m := m[p \mapsto 1]$;

  if $m \not\models (q_1 \land .. \land q_k \Rightarrow \bot) \in Gs$ **then** **return** *unsat* ;

return m

Exercise 8.11

Solve

$$\{\top \Rightarrow p, \quad q \land r \land t \Rightarrow p, \quad p \Rightarrow q, \quad p \land r \Rightarrow t,$$
$$\neg p \land q \Rightarrow t, \quad r \Rightarrow \bot, \quad p \land q \land t \Rightarrow \bot\}$$

Recognizing Horn clauses

Sometimes a set of clauses are not immediately recognizable as Horn clause.

We may convert a CNF into a Horn formula by flipping the negation sign for some variables. Such CNF are called Horn clause renameable.

Definition 8.7

Let F be a CNF formula and m be a model. Let $flip(F, m)$ denote the formula obtained by flipping the variables that are assigned 1 in m.

Example 8.9

$flip((p \lor \neg q \lor \neg s), \{p \mapsto 1, q \mapsto 0, s \mapsto 1, ..\}) = (\neg p \lor \neg q \lor s)$

Exercise 8.12

Calculate $flip((\neg p \lor q \lor \neg s), \{p \mapsto 1, q \mapsto 1, s \mapsto 0, ..\})$

Renaming Horn clauses

Theorem 8.10

A CNF formula $F = \{C_1, \dots, C_n\}$, where $C_i = \{\ell_{i1}, \dots, \ell_{i|C_i|}\}$ is Horn clause renameable iff the following 2-SAT formula is satisfiable.[*]

$$G = \{\ell_{ij} \vee \ell_{ik} | i \in 1..n \text{ and } 1 \leq j < k \leq |C_i|\}$$

Proof.

Forward direction: there is a model m such that $flip(F, m)$ is a Horn formula

claim: $m \models G$

consider a clause $\ell_{ij} \vee \ell_{ik} \in G$

case $\ell_{ij} = p, \ell_{ik} = q$: one of them must flip,i.e.,$m(p) = 1$ or $m(q) = 1$

case $\ell_{ij} = \neg p, \ell_{ik} = \neg q$: at least one must not flip, i.e., not $m(p) = m(q) = 1$

case $\ell_{ij} = \neg p, \ell_{ik} = q$: if p flips then q must,i.e., if $m(p) = 1$ then $m(q) = 1$

In all the three cases $m \models \ell_{ij} \vee \ell_{ik}$.

[*] H. Lewis. Renaming a Set of Clauses as a Horn Set. J. of the ACM, 25:134-135, 1978.

Renaming Horn clauses(contd.)

Proof(contd.)

Reverse direction: Let $m \models G$. Let $F' = \mathit{flip}(F, m)$.

claim: F' is a Horn formula

Suppose F' is not a Horn formula.

Then, there are positive literals ℓ'_{ij} and ℓ'_{ik} in clause C_i in F'.

Therefore, $m \not\models \ell_{ij} \vee \ell_{ik\text{(why?)}}$. Contradiction. $\square$

Exercise 8.13

What is the complexity of checking if a formula is Horn clause renameable?

Exercise 8.14

Can you improve the above complexity?

Topic 8.5

Problems

Unsat XOR-sat

Exercise 8.15

Give an unsat XOR-sat formula that has only xors with more than three arguments.

Unsat 2-CNF**

Exercise 8.16

Let us suppose we have n variables in a 2-CNF problem. What is the maximum number of clauses in the formula such that the formula is satisfiable?

Unsatisfiable core of 2-CNF

Exercise 8.17

An unsatisfiable core of an unsatisfiable CNF formula is a (preferably minimal) subset of the formula that is also unsatisfiable. Give an algorithm to compute a minimal unsatisfiable core of 2-CNF formula.

Horn SAT Truth to false

Exercise 8.18

In the Horn solving algorithm, we started with all false assignment and incrementally turned the variables true.
a. Give an modified algorithm that starts with all true initial assignment and finds satisfying assignment for the Horn clauses.
b. Can we also start with any initial assignment? If yes, how the algorithm needs to be modified?

Topic 8.6

Extra slides: Single look ahead unit resolution

Single lookahead unit resolution(SLUR)

This subclass is defined using the following algorithm.

Algorithm 8.2: $\mathrm{SLUR}(F)$

Input: F: CNF formula
Output: model/unknown

$m = \lambda x.\mathtt{null}$;
while *m is partial* **do**
  Choose an unassigned variable p in m ;
  Apply unit clause propagation and extend $m[p \mapsto 1]$ to m';
  if $m' \not\models F$ **then**
    Apply unit clause propagation and extend $m[p \mapsto 0]$ to m';
    if $m' \not\models F$ **then** **return** *unknown* ;
  $m := m'$;
return m

Definition 8.8

F belongs to SLUR class *if* $\mathrm{SLUR}(F)$ *can never return* unknown.

SLUR recognition

There is no efficient way to recognize a SLUR formula.

More low complexity classes

- Extended horn
- CC balanced
- q-Horn
- 2-SAT linear
- Horn linear
- Linear autarky
- add others

Commentary: Recognition of q-Horn formulas in linear time. Endre Boros, Hammer, and Sun

k-BRLR class

Exercise 8.19

A CNF formula is in k-BRLR if all consequence derived from it using resolution have at most k literals. What is the complexity of checking satisfiability of formulas in k-BRLR?

Another class

Definition 8.9

Define cc-balanced formula
Algorithms for the Satisfiability (SAT) Problem - Gu et. al. p67-74

Exercise 8.20

Consider a CNF formula F such that every clause in F has at least two literals and for each variable p there is a model of F with p is true and a model with p is false. Give a linear time algorithm to find a satisfying assignment of F.

End of Lecture 8