CS228 Logic for Computer Science 2020

Lecture 15: First-order logic - Semantics

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Topic 15.1

FOL - semantics

Semantics : structures

Definition 15.1

For signature $\mathbf{S} = (\mathbf{F}, \mathbf{R})$*, a* **S***-structure* m *is a*

$$(D_m; \{f_m : D_m^n \to D_m | f/n \in \mathbf{F}\}, \{P_m \subseteq D_m^n | P/n \in \mathbf{R}\}),$$

where D_m *is a nonempty set. Let* **S***-Mods denotes the set of all* **S***-structures.*

Some terminology

- D_m is called domain of m.
- f_m assigns meaning to f under structure m.
- Similarly, P_m assigns meaning to P under structure m.

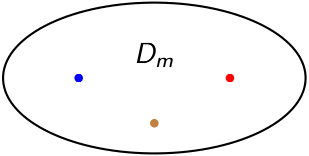
Commentary: Structures are also known as interpretations/models.

Example: structure

Example 15.1

Consider $\mathbf{S} = (\{c/0, f/1, g/2\}, \{H/1, M/2\})$.

Let us suppose our structure m *has domain* $D_m = \{\textcolor{red}{\bullet}, \textcolor{blue}{\bullet}, \textcolor{brown}{\bullet}\}$.



We need to assign value to each function.

- $c_m = \textcolor{blue}{\bullet}$
- $f_m = \{\textcolor{blue}{\bullet} \mapsto \textcolor{brown}{\bullet}, \textcolor{brown}{\bullet} \mapsto \textcolor{red}{\bullet}, \textcolor{red}{\bullet} \mapsto \textcolor{red}{\bullet}\}$
- $g_m = \{(\textcolor{blue}{\bullet}, \textcolor{blue}{\bullet}) \mapsto \textcolor{brown}{\bullet}, (\textcolor{blue}{\bullet}, \textcolor{brown}{\bullet}) \mapsto \textcolor{blue}{\bullet}, (\textcolor{blue}{\bullet}, \textcolor{red}{\bullet}) \mapsto \textcolor{red}{\bullet},$
  $(\textcolor{red}{\bullet}, \textcolor{blue}{\bullet}) \mapsto \textcolor{brown}{\bullet}, (\textcolor{red}{\bullet}, \textcolor{brown}{\bullet}) \mapsto \textcolor{brown}{\bullet}, (\textcolor{red}{\bullet}, \textcolor{red}{\bullet}) \mapsto \textcolor{brown}{\bullet},$
  $(\textcolor{brown}{\bullet}, \textcolor{blue}{\bullet}) \mapsto \textcolor{red}{\bullet}, (\textcolor{brown}{\bullet}, \textcolor{brown}{\bullet}) \mapsto \textcolor{brown}{\bullet}, (\textcolor{brown}{\bullet}, \textcolor{red}{\bullet}) \mapsto \textcolor{blue}{\bullet}\}$

We also need to assign values to each predicate.

- $H_m = \{\textcolor{blue}{\bullet}, \textcolor{brown}{\bullet}\}$
- $M_m = \{(\textcolor{blue}{\bullet}, \textcolor{blue}{\bullet}), (\textcolor{brown}{\bullet}, \textcolor{red}{\bullet})\}$

Exercise 15.1

a. How many structure are there for the signature with the above domain?

b. Suppose $P/0 \in \mathbf{R}$*, give a value to* P_m.

Example: structure

Example 15.2

Consider $\mathbf{S} = (\{\cup/2\}, \{\in /2\})$.

$m = (\mathbb{N}; \cup_m = max, \in_m = \{(i,j) | i < j\})$ *is a* **S***-structure.*

Semantics: assignments

Recall, We also have variables. Who will assign to the variables?

Definition 15.2

An assignment is a map $\nu : \mathbf{Vars} \to D_m$

Example 15.3

In our running example the domain is $\mathbb{N}$. *We may have the following assignment.*

$$\nu = \{x \mapsto 2, y \mapsto 3\}$$

Semantics: term value

Definition 15.3

For a structure m and assignment ν, we define $m^\nu : T_{\mathbf{S}} \rightarrow D_m$ as follows.

$$
\begin{aligned}
m^\nu(x) &\triangleq \nu(x) && x \in \mathbf{Vars} \\
m^\nu(f(t_1, \ldots, t_n)) &\triangleq f_m(m^\nu(t_1), \ldots, m^\nu(t_n))
\end{aligned}
$$

Definition 15.4

Let t be a closed term. $m(t) \triangleq m^\nu(t)$ *for any* ν.

Example 15.4

Consider assignment $\nu = \{x \mapsto 2, y \mapsto 3\}$ *and term* $\cup(x, y)$*.*
$m^\nu(\cup(x, y)) = max(2, 3) = 3$

Example: satisfiability

Example 15.5

Consider $\mathbf{S} = (\{s/1, +/2\}, \{\})$ *and term* $s(x) + y$

Consider structure $m = (\mathbb{N}; succ, +^{\mathbb{N}})$ *and assignment* $\nu = \{x \mapsto 3, y \mapsto 2\}$

$$m^{\nu}(s(x) + y) = m^{\nu}(s(x)) +^{\mathbb{N}} m^{\nu}(y) = succ(m^{\nu}(x)) +^{\mathbb{N}} 2 = succ(3) +^{\mathbb{N}} 2 = 6$$

Semantics: satisfaction relation

Definition 15.5

We define the satisfaction relation $\models$ among structures, assignments, and formulas as follows

- $m, \nu \models \top$
- $m, \nu \models P(t_1, \dots, t_n)$ *if* $(m^\nu(t_1), \dots, m^\nu(t_n)) \in P_m$
- $m, \nu \models t_1 = t_2$ *if* $m^\nu(t_1) = m^\nu(t_n)$
- $m, \nu \models \neg F$ *if* $m, \nu \not\models F$
- $m, \nu \models F_1 \lor F_2$ *if* $m, \nu \models F_1$ *or* $m, \nu \models F_2$

  skipping other boolean connectives
- $m, \nu \models \exists x.(F)$ *if* *there is* $u \in D_m : m, \nu[x \mapsto u] \models F$
- $m, \nu \models \forall x.(F)$ *if* *for each* $u \in D_m : m, \nu[x \mapsto u] \models F$

Example: satisfiability

Example 15.6

Consider $\mathbf{S} = (\{s/1, +/2\}, \{\})$ *and formula* $\exists z. s(x) + y = s(z)$

Consider structure $m = (\mathbb{N}; succ, +^{\mathbb{N}})$ *and assignment* $\nu = \{x \mapsto 3, y \mapsto 2\}$

We have seen $m^{\nu}(s(x) + y) = 6$.

$m^{\nu[z \mapsto 5]}(s(x) + y) = m^{\nu}(s(x) + y) = 6.$ *//Since z does not occur in the term*

$m^{\nu[z \mapsto 5]}(s(z)) = 6$

Therefore, $m, \nu[z \mapsto 5] \models s(x) + y = s(z)$.

$m, \nu \models \exists z. s(x) + y = s(z)$.

Exercise: satisfaction relation

Exercise 15.2

Consider sentence $F = \exists x.\forall y.\ \neg y \in x$ *(what does it say to you!)*

Consider $m = (\mathbb{N}; \cup_m = max, \in_m = \{(i,j) | i < j\})$ *and* $\nu = \{x \mapsto 2, y \mapsto 3\}$.

Does $m, \nu \models F$*?*

Exercise: structure

Consider $\mathbf{S} = (\{c/0, f/1\}, \{H/1, M/2\})$. Let us suppose structure m has $D_m = \{{\color{red}\bullet}, {\color{blue}\bullet}, {\color{brown}\bullet}\}$ and the values of the symbols in m are

- $c_m = {\color{blue}\bullet}$
- $f_m = \{{\color{blue}\bullet} \mapsto {\color{brown}\bullet}, {\color{brown}\bullet} \mapsto {\color{red}\bullet}, {\color{red}\bullet} \mapsto {\color{red}\bullet}\}$
- $H_m = \{{\color{blue}\bullet}, {\color{brown}\bullet}\}$
- $M_m = \{({\color{blue}\bullet}, {\color{blue}\bullet}), ({\color{brown}\bullet}, {\color{red}\bullet})\}$

Exercise 15.3

Is the following sentence satisfied by the above structure?

- $\exists x.H(x)$
- $\exists x.H(f(x))$
- $\forall x.H(x)$
- $H(c)$

Attendance quiz

Is following sentence satisfied by the structure on the slide?

$c = f(f(c))$
$\exists x. H(x) \wedge H(f(x))$
$\exists x. M(x, x)$
$\exists x. \neg M(x, x)$
$\exists x. M(x, f(x))$
$\exists x. \neg M(x, f(x))$
$\forall x. (H(x) \Rightarrow f(x) \neq c)$
$\forall x. (c = x \Rightarrow H(x))$
$\forall x. (H(x) \Rightarrow \exists y. M(x, y))$

$c = f(c)$
$\exists x. H(f(f(x)))$
$\forall x. M(x, x)$
$\forall x. \neg M(x, x)$
$\forall x. M(x, f(x))$
$\forall x. \neg M(x, f(x))$
$\exists x. (H(x) \wedge f(x) = c)$
$\exists x. (c = x \wedge \neg H(x))$

$\exists x. (H(x) \wedge \forall y. \neg M(x, y))$

Why nonempty domain?

We are required to have nonempty domain in the structure. Why?

Example 15.7

Consider formula $\forall x.(H(x) \land \neg H(x))$.

Should any structure satisfy the formula? *Noooooooo..*

But, if we allow $m = \{\emptyset; H_m = \emptyset\}$ *then*

$$m \models \forall x.(H(x) \land \neg H(x)).$$

Due to this counterintuitive behavior, the empty domain is disallowed.

Example: non-standard structures

Example 15.8

Consider $\mathbf{S} = (\{\mathbf{0}/0, s/1, +/2\}, \{\})$ *and formula* $\exists z. s(x) + y = s(z)$

Unexpected structure: *Let* $m = (\{a, b\}^*; \epsilon, append\_a, concat)$.

- *The domain of* m *is the set of all strings over alphabet* $\{a, b\}$.
- $append\_a$: *appends a in the input and*
- $concat$: *joins two strings.*

Let $\nu = \{x \mapsto ab, y \mapsto ba\}$.
Since $m, \nu[z \mapsto abab] \models s(x) + y = s(z)$,
$m, \nu \models \exists z. s(x) + y = s(z)$.

Exercise 15.4

- *Show* $m, \nu[y \mapsto bb] \not\models \exists z. s(x) + y = s(z)$
- *Give an assignment* ν *s.t.* $m, \nu \models x \neq 0 \Rightarrow \exists y.\ x = s(y)$.
  Show $m \not\models \forall x.\ x \neq 0 \Rightarrow \exists y.\ x = s(y)$.

Satisfiable, true, valid, and unsatisfiable

We say

- F is *satisfiable* if there are m and ν such that $m, \nu \models F$
- Otherwise, F is called unsatisfiable
- F is *true* in m ($m \models F$) if for all ν we have $m, \nu \models F$
- F is *valid* ($\models F$) if for all ν and m we have $m, \nu \models F$

If F is a sentence, ν has no influence in the satisfaction relation.(why?)

For sentence F, we say

- F is *true* in m if $m \models F$
- Otherwise, F is *false* in m.

Extended satisfiability

We extend the usage of $\models$.

Definition 15.6

Let Σ be a (possibly infinite) set of formulas.
$m, \nu \models \Sigma$ if $m, \nu \models F$ for each $F \in \Sigma$.

Definition 15.7

Let M be a (possibly infinite) set of structures.
$M \models F$ if for each $m \in M$, $m \models F$.

Implication and equivalence

Definition 15.8

Let Σ be a (possibly infinite) set of formulas.
$\Sigma \models F$ *if for each structure m and assignment ν if $m, \nu \models \Sigma$ then $m, \nu \models F$.*

$\Sigma \models F$ is read Σ implies F. If $\{G\} \models F$ then we may write $G \models F$.

Definition 15.9

Let $F \equiv G$ *if $G \models F$ and $F \models G$.*

Equisatisfiable and equivalid

Definition 15.10
Formulas F and G are equisatisfiable *if*

$$F \text{ is sat} \quad \text{iff} \quad G \text{ is sat.}$$

Definition 15.11
Formulas F and G are equivalid *if*

$$\models F \quad \text{iff} \quad \models G.$$

Topic 15.2

Problems

FOL to PL

Exercise 15.5

Give the restrictions on FOL such that it becomes the propositional logic. Give an example of FOL model of a non-trivial propositional formula.

Valid formulas

Exercise 15.6

Prove/Disprove the following formulas are valid.

- $\forall x.P(x) \Rightarrow P(c)$
- $\forall x.(P(x) \Rightarrow P(c))$
- $\exists x.(P(x) \Rightarrow \forall x.P(x))$
- $\exists y \forall x.R(x,y) \Rightarrow \forall x \exists y.R(x,y)$
- $\forall x \exists y.R(x,y) \Rightarrow \exists y \forall x.R(x,y)$

Distributively

Exercise 15.7

Show the validity of the following formulas.

1. $\neg\forall x.\ P(x) \Leftrightarrow \exists x.\ \neg P(x)$
2. $(\forall x.\ (P(x) \land Q(x))) \Leftrightarrow \forall x.\ P(x) \land \forall x.\ Q(x)$
3. $(\exists x.\ (P(x) \lor Q(x))) \Leftrightarrow \exists x.\ P(x) \lor \exists x.\ Q(x)$

Show $\forall$ does not distribute over $\lor$.

Show $\exists$ does not distribute over $\land$.

Encode mod k

Exercise 15.8

Give an FOL sentence that encodes that there are n elements in any satisfying structure, such that $n \mod k = 0$ for a given k.

Hierarchy of formulas

Exercise 15.9

Topic 15.3

Extra slides: some properties of models

Homomorphisms of models

Definition 15.12

Consider $\mathbf{S} = (\mathbf{F}, \mathbf{R})$*. Let* m *and* m' *be* $\mathbf{S}$*-models.*
A function $h : D_m \to D_{m'}$ *is a homomorphism of* m *into* m' *if the following holds.*

- *for each* $f/n \in \mathbf{F}$*, for each* $(d_1, .., d_n) \in D_m^n$

$$h(f_m(d_1, .., d_n)) = f_{m'}(h(d_1), .., h(d_n))$$

- *for each* $P/n \in \mathbf{R}$*, for each* $(d_1, .., d_n) \in D_m^n$

$$(d_1, .., d_n) \in P_m \quad \textit{iff} \quad (h(d_1), .., h(d_n)) \in P_{m'}$$

Definition 15.13

A homomorphism h *of* m *into* m' *is called isomorphism if* h *is one-to-one.*
m *and* m' *are called isomorphic if an* h *exists that is also onto.*

Example : homomorphism

Example 15.9

Consider $\mathbf{S} = (\{+/2\}, \{\})$.

Consider $m = (\mathbb{N}, +^{\mathbb{N}})$ *and* $m = (\mathcal{B}, \oplus^{\mathcal{B}})$,

$h(n) = n \text{ mod } 2$ *is a homomorphism of* m *into* m'.

Homomorphism theorem for terms and quantifier-free formulas without $=$

Theorem 15.1

Let h be a homomorphism of m into m'. Let ν be an assignment.

1. *For each term t, $h(m^{\nu}(t)) = m'^{(\nu \circ h)}(t)$*
2. *If formula F is quantifier-free and has no symbol "$=$"*

$$m^{\nu} \models F \quad \textit{iff} \quad m'^{(\nu \circ h)} \models F$$

Proof.

Simple structural induction. $\square$

Exercise 15.10

For a quantifier-free formula F that may have symbol "$=$", show

$$\textit{if} \quad m^{\nu} \models F \quad \textit{then} \quad m'^{(\nu \circ h)} \models F$$

Why the reverse direction does not work?

Homomorphism theorem with $=$

Theorem 15.2

Let h be a homomorphism of m into m'. Let ν be an assignment. If h is isomorphism then the reverse implication also holds for formulas with "$=$".

Proof.

Let us suppose $m'^{(\nu \circ h)} \models s = t$.
Therefore, $m'^{(\nu \circ h)}(s) = m'^{(\nu \circ h)}(t)$.
Therefore, $h(m^{\nu}(s)) = h(m^{\nu}(t))$.
Due to the one-to-one condition of h, $m^{\nu}(s) = m^{\nu}(t)$.
Therefore, $m^{\nu} \models s = t$. $\square$

Exercise 15.11

For a formula F (with quantifiers) without symbol "$=$", show

$$\text{if} \quad m'^{(\nu \circ h)} \models F \quad \text{then} \quad m^{\nu} \models F.$$

Why the reverse direction does not work?

Commentary: Note that that implication direction is switched from the previous exercise.

Homomorphism theorem with quantifiers

Theorem 15.3

Let h be a isomorphism of m into m' and ν be an assignment.
If h is also onto, the reverse direction also holds for the quantified formulas.

Proof.

Let us assume, $m^{\nu} \models \forall x.F$.
Choose $d' \in D_{m'}$.
Since h is onto, there is a d such that $d = h(d')$.
Therefore, $m^{\nu[x \mapsto d]} \models F$.
Therefore, $m'^{\nu[x \mapsto d']} \models F$.
Therefore, $m'^{(\nu \circ h)} \models \forall x.\ F$. □

Theorem 15.4

If m and m' are isomorphic then for all sentences F,

$$m \models F \quad \textit{iff} \quad m' \models F.$$

Commentary: The reverse direction of the above theorem is not true.

End of Lecture 15