CS228 Logic for Computer Science 2020

Lecture 16: FOL - Understanding FOL

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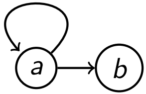
Topic 16.1

Example formulas

Example: graph structures

Example 16.1

Consider $\mathbf{S} = (\{\}, \{E/2\})$ *and* $m = (\{a, b\}; \{(a, a), (a, b)\})$*.*
m may be viewed as the following graph.



$$m, \{x \to a\} \models E(x, x) \land \exists y.(E(x, y) \land \neg E(y, y))$$

Exercise 16.1

Give another structure and assignment that satisfies the above formula

Example : counting

Example 16.2

Consider $\mathbf{S} = (\{\}, \{E/2\})$

The following sentence is false in all the structures with one element domain

$$\forall x. \neg E(x,x) \land \exists x \exists y. E(x,y)$$

Exercise: counting

Exercise 16.2

Give a sentence that is true only in the structures with more than two elements

Exercise 16.3

a. Give a sentence that is true only in infinite structures

b. Does only finite structures satisfy the negation of the sentence in (a)?

Exercise 16.4

a. Give a sentence that is true only in structures with less than or equal to two element domains.

b. Can you answer (a) without using $=$?

Topic 16.2

Substitution

Substitution

Definition 16.1

A substitution σ is a map from $\mathbf{Vars} \rightarrow T_{\mathbf{S}}$. We will write $t\sigma$ to denote $\sigma(t)$.

Definition 16.2

We say σ has finite support if only finite variables do not map to themselves. σ with finite support is denoted by $[t_1/x_1, .., t_n/x_n]$ or $\{x_1 \mapsto t_1, .., x_n \mapsto t_n\}$.

Commentary: We have seen substitution in propositional logic. Now in FOL, substitution is not a simple matter. Before introducing the other aspects of FOL. Let us present substitution first.

Substitution on terms

Definition 16.3

For $t \in T_{\mathbf{S}}$, let the following naturally define $t\sigma$ as extension of σ.

- $c\sigma \triangleq c$
- $(f(t_1, \dots, t_n))\sigma \triangleq f(t_1\sigma, \dots, t_n\sigma)$

Example 16.3

Consider $\sigma = [f(x, y)/x, f(y, x)/y]$

- $x\sigma = f(x, y)$
- $f(x, y)\sigma = f(f(x, y), f(y, x))$
- $(f(x, y)\sigma)\sigma =?$

Composition

Definition 16.4

Let σ_1 and σ_2 be substitutions. The composition $\sigma_1\sigma_2$ of the substitutions is defined as follows.

$$\text{For each } x \in \textbf{Vars},\ x(\sigma_1\sigma_2) \triangleq (x\sigma_1)\sigma_2.$$

Theorem 16.1

For each $t \in T_{\mathbf{S}}$, $t(\sigma_1\sigma_2) = (t\sigma_1)\sigma_2$

Proof.

Proved by trivial structural induction. □

Theorem 16.2

$\sigma_1(\sigma_2\sigma_3) = (\sigma_1\sigma_2)\sigma_3$ *(associativity)*

Proof.

Consider variable x.

$(x\sigma_1)(\sigma_2\sigma_3) = ((x\sigma_1)\sigma_2)\sigma_3 = (x(\sigma_1\sigma_2))\sigma_3$ □

Subtituion on atoms

We further extend the substitution σ to atoms.

Definition 16.5

For $F \in A_{\mathbf{S}}$, $F\sigma$ is defined as follows.

- $\top\sigma \triangleq \top$
- $\bot\sigma \triangleq \bot$
- $P(t_1, \dots, t_n)\sigma \triangleq P(t_1\sigma, \dots, t_n\sigma)$
- $(t_1 = t_2)\sigma \triangleq t_1\sigma = t_2\sigma$

Theorem 16.3

For each $F \in A_{\mathbf{S}}$, $F(\sigma_1\sigma_2) = (F\sigma_1)\sigma_2$

Proof.

Proved by trivial structural induction. □

Substitution

Sometimes, we may need to remove variable x from the support of σ.

Definition 16.6

Let $\sigma_x = \sigma[x \mapsto x]$*.*

Example 16.4

Consider $\sigma = [f(x,y)/x, f(y,x)/y]$

$\sigma_x = [f(y,x)/y]$

Commentary: The need of the definition will be clear soon.

Substitution in formulas (Incorrect)

Now we extend the substitution σ to all the formulas.

Definition 16.7

For $F \in \mathbf{P_S}$, $F\sigma$ is defined as follows.

- $(\neg G)\sigma \triangleq \neg(G\sigma)$
- $(G \circ H)\sigma \triangleq (G\sigma) \circ (H\sigma)$ *for some binary operator* $\circ$
- $(\forall x.G)\sigma \triangleq \forall x.(G\sigma_x)$
- $(\exists x.G)\sigma \triangleq \exists x.(G\sigma_x)$

Example 16.5

- $(P(x) \Rightarrow \forall x.Q(x))[y/x] = (P(y) \Rightarrow \forall x.Q(x))$
- $(\exists y.\ x \neq y)[z/x] = (\exists y.\ z \neq y)$
- $(\exists y.\ x \neq y)[y/x] = (\exists y.\ y \neq y)$ ☹ *Undesirable!!!*

Some substitutions should be disallowed.

Commentary: The above näive definition of the substitution in formulas is incorrect. In the next slide, we present the correct definition.

Substitution in formulas(Correct)

Definition 16.8

σ is suitable wrt to formula G and variable x if for all $y \neq x$, if y occurs in G then x does not occur in $y\sigma$.

Now we correctly extend the substitution σ to all formulas.

Definition 16.9

For $F \in \mathbf{P_S}$, $F\sigma$ is defined as follows.

- $(\neg G)\sigma \triangleq \neg(G\sigma)$
- $(G \circ H)\sigma \triangleq (G\sigma) \circ (H\sigma)$ *for some binary operator* $\circ$
- $(\forall x.G)\sigma \triangleq \forall x.(G\sigma_x)$, *where σ is suitable wrt G and x*
- $(\exists x.G)\sigma \triangleq \exists x.(G\sigma_x)$, *where σ is suitable wrt G and x*

It is not a true restriction. We will see later.

For short hand, we may write a formula as $F(x_1, \dots, x_k)$, where we say that $x_1, \dots, x_k$ are the variables that play a special role in the formula F. Let $F(t_1, \dots, t_n)$ be $F[t_1/x_1, \dots, t_n/x_n]$.

Substitution composition

Theorem 16.4

if $F\sigma_1$ *and* $(F\sigma_1)\sigma_2$ *are defined then* $(F\sigma_1)\sigma_2 = F(\sigma_1\sigma_2)$

Proof.

We prove it by induction. Non-quantifier cases are simple structural induction.

Assume $F = \forall x.G$

Since $F\sigma_1$ is defined, $G\sigma_{1x}$ is defined.

Since $(F\sigma_1)\sigma_2$ is defined, $(G\sigma_{1x})\sigma_{2x}$ is defined (why?).

By Ind. Hyp. $(G\sigma_{1x})\sigma_{2x} = G(\sigma_{1x}\sigma_{2x})$

claim: $G(\sigma_{1x}\sigma_{2x}) = G(\sigma_1\sigma_2)_x$

Choose $y \in FV(G)$ and $y \neq x$

$y(\sigma_{1x}\sigma_{2x})$

$= ((y\sigma_{1x})\sigma_{2x}) = ((y\sigma_1)\sigma_{2x})$

$= ((y\sigma_1)\sigma_2)$ $\quad x \notin FV(y\sigma_1)$ (why?)

$= y(\sigma_1\sigma_2) = y(\sigma_1\sigma_2)_x$

$(\forall x.G\sigma_1)\sigma_2 = (\forall x.G(\sigma_{1x}\sigma_{2x})) = (\forall x.G(\sigma_1\sigma_2)_x) = F(\sigma_1\sigma_2)$ □

Formal rules for FOL

- The old rules will continue to work
- We need new rules for..... quantifiers
- Let us see how do we develop those!

∃-Intro Quantifiers

If some fact is true about some term, we can introduce the $\exists$

$$\exists - \textsc{Intro}\frac{\Sigma \vdash F(t)}{\Sigma \vdash \exists y.\ F(y)}\ y \notin \mathbf{Vars}(F(z))$$

Example 16.6

1. $\{H(x)\} \vdash H(x)$ *Assumption*
2. $\{H(x)\} \vdash \exists y.\ H(y)$ *$\exists$-Intro applied to 1*

Example 16.7

1. $\{\exists x.\ c \neq x\} \vdash \exists x.\ c \neq x$ *Assumption*
2. $\{\exists x.\ c \neq x\} \vdash \exists x.\ \exists x.\ x \neq x$ *$\exists$-Intro applied to 1*✗

We have the side condition to avoid invalid substitutions

More bad derivations

Example 16.8

1. $\{x = 1, y = 2\} \vdash x \neq y$ *...from some reasoning...*
2. $\{x = 1, y = 2\} \vdash \exists y.\ y \neq y$ *$\exists$-Intro applied to 1*✗

We have the side condition to avoid invalid substitutions

Example 16.9

1. $\Sigma \vdash F(f(x), x)$ *Premise*
2. $\Sigma \vdash \exists x. F(x, x)$ *$\exists$-Intro applied to 1*✗

We have the side condition to avoid invalid substitutions

How to intro $\forall$?

We have seen the following proof in our life.

- Consider a fresh name x to represent a number.
- We prove $Fact(x)$
- We conclude $\forall x.Fact(x)$.

∀-Intro for variables

If something is true about a variable that is not referred elsewhere.

Then it must be true for any value in the universe.

No reference condition

$$\forall - \textsc{Intro}\frac{\Sigma \vdash F(x)}{\Sigma \vdash \forall y.\ F(y)}\ y \notin \mathbf{Vars}(F(z)),\ x \in \mathbf{Vars},\ \text{and}\ x \notin \mathbf{Vars}(\Sigma)$$

Example 16.10

1. $\{H(x)\} \vdash H(x)$ *Assumption*
2. $\{H(x)\} \vdash \forall y.\ H(y)$ *∀-Intro applied to 1*✗

Since x is referred in left hand side, the above derivation is wrong.

∀-Intro (for constants)

Constants may play the similar role

$$\forall - \textsc{Intro} \frac{\Sigma \vdash F(c)}{\Sigma \vdash \forall y.\ F(y)} y \not\in \textbf{Vars}(F),\ c \text{ not referred in } \Sigma,\ \text{and } c/0 \in \textbf{F}$$

Example 16.11

1. $\Sigma \vdash H(c)$ *Premise and $c \not\in \textbf{Vars}(\Sigma)$*
2. $\Sigma \vdash \forall y.\ H(y)$ *∀-Intro applied to 1*

End of Lecture 16