CS228 Logic for Computer Science 2020

Lecture 19: FOL - conjunctive normal form

Instructor: Ashutosh Gupta

IITB, India

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Topic 19.1

First-order logic conjunctive normal form (FOL CNF)

CNF normalization steps

Any modern FOL solver first convert sentences into CNF.

The following transformations results in the CNF.

1. Rename apart : rename variables for each quantifier
2. Prenex : bringing quantifiers to front
3. Skolemization: remove existential quantifiers (only sat preserving)
4. CNF transformation: turn the quantifier-free part of the sentence into CNF
5. Syntactical removal of universal quantifiers: a CNF with free variables.

Step 1: rename apart

Definition 19.1

A formula F is renamed apart if no quantifier in F use a variable that is used by another quantifier or occurs as free variable in F.

Due to the theorems like the following, we can assume that every quantifier has different variable. If not, we can rename quantified variables apart.

Theorem 19.1

Let F is a **S***-formulas and y does not occur in F.*

$$\forall x.F(x) \equiv \forall y.F(y)$$

Example 19.1

Consider formula $\neg(\exists x.\forall y R(x, y) \Rightarrow \forall y.\exists x R(x, y))$
After renaming apart we obtain the following

$$\neg(\exists x.\forall y R(x, y) \Rightarrow \forall z.\exists w R(w, z))$$

Step 2: prenex form

Definition 19.2

A formula F is in prenex form if all the quantifiers of the formula occur as prefix of F. The quantifier-free suffix of F is called matrix of F.

Due to the following equivalences, we move quantifiers to the front.

- $\neg(\exists x.F) \equiv \forall x.\neg F$
- $\neg(\forall x.F) \equiv \exists x.\neg F$
- $\forall x.F \land G \equiv \forall x.(F \land G)$
- $\exists x.F \land G \equiv \exists x.(F \land G)$
- $\forall x.F \lor G \equiv \forall x.(F \lor G)$
- $\exists x.F \lor G \equiv \exists x.(F \lor G)$
- $\forall x.F \Rightarrow G \equiv \exists x.(F \Rightarrow G)$
- $\exists x.F \Rightarrow G \equiv \forall x.(F \Rightarrow G)$
- $F \Rightarrow \forall x.G \equiv \forall x.(F \Rightarrow G)$
- $F \Rightarrow \exists x.G \equiv \exists x.(F \Rightarrow G)$

Example: prenex form

Exercise 19.1

We convert $\neg(\exists x.\forall y.R(x,y) \Rightarrow \forall z.\exists w.R(w,z))$ *into prenex form as follows*

- $\neg\forall z.(\exists x.\forall y.R(x,y) \Rightarrow \exists w.R(w,z))$
- $\neg\forall z.\exists w.(\exists x.\forall y.R(x,y) \Rightarrow R(w,z))$
- $\neg\forall z.\exists w.\forall x.\exists y.(R(x,y) \Rightarrow R(w,z))$
- $\underbrace{\exists z.\forall w.\exists x.\forall y.}_{\text{Quantifiers}}\underbrace{\neg(R(x,y) \Rightarrow R(w,z))}_{\text{body/matrix of the formula}}$

We move quantifier forward step by step.

Step 3: Skolemization

Skolemization removes existential quantifiers from the prenex sentence and only the universal quantifiers are left.

Example 19.2

Let us suppose. We know "for every man there is a woman".

$$\forall m.\ \exists w.\ Relationship(m, w)$$

To satisfy the sentence, we need to find a woman for each man.

In other words, there is a function $f: Men \rightarrow Women$.

In terms of FOL, we may write

$$\forall m. Relationship(m, f(m))$$

The replacement of $\exists$ by a function is called skolemization. And f are called skolem functions.

Introduction of skolem function with free variables

Theorem 19.2

Let F be a $\mathbf{S}$-formula, $FV(F) = \{x, y_1, \dots, y_n\}$ and $f/n \in \mathbf{F}$ does not occur in F. For each structure m', there is a structure m such that

$$m \models \exists x.F(x) \Rightarrow F(f(y_1, \dots, y_n)).$$

and m and m' only differ on interpretation of f.

Proof.

Consider a structure m'. Let $f' := f_{m'}$. Let us construct m as follows.

Since F has free variables, we need to consider assignments.
Consider assignment ν such that $\nu(y_1) = d_1, .., \nu(y_n) = d_n$, $d_i \in D_{m'}$.
  case $m', \nu \models \exists x.F$: there is $d \in D_{m'}$ such that $m', \nu[x \to d] \models F$.
  case $m', \nu \not\models \exists x.F$: choose any $d \in D_{m'}$.

$f' := f'[(d_1, .., d_n) \mapsto d]$.
Let us define $m \triangleq m'[f \mapsto f']$. ...

Introduction of skolem function with free variables(contd.)

Proof(contd.)

Since f does not occur in F, if $m, \nu \models \exists x.F$ then $m', \nu \models \exists x.F$.
Due to construction of m, $m, \nu \models F\{x \mapsto f(y_1, \dots, y_n)\}$(why?). □

Exercise 19.2

Show there is m s.t. $m \models F\{x \mapsto f(y_1, \dots, y_n)\} \Rightarrow \forall x.F$

Introduction of skolem functions under quantifiers

Theorem 19.3

Let $F(x)$ be a $\mathbf{S}$-formula with $FV(F) = \{x, y_1, \dots, y_n\}$. Let $f/n \in \mathbf{F}$ s.t. f does not occur in $F(x)$.

$$\forall y_1, ...y_n.\exists x.F(x) \text{ is sat} \quad \text{iff} \quad \forall y_1, ...y_n.\exists x.F(f(y_1, \dots, y_n)) \text{ is sat}$$

Proof.

Reverse direction

If $m, \nu \models F(f(y_1, \dots, y_n))$ then there is a d for every $\nu(y_1), \dots, \nu(y_n)$ such that $m, \nu[x \mapsto f_m(\nu(y_1), ..\nu(y_n))] \models F(x)$.

Therefore, $m, \nu \models \exists x.F(x)$.

Therefore, $\models F(f(y_1, \dots, y_n)) \Rightarrow \exists x.F(x)$.

Due to our proof system,

$\models \forall y_1, ...y_n.F(f(y_1, \dots, y_n)) \Rightarrow \forall y_1, ...y_n.\exists x.F(x)$. ...

Introduction of skolem functions under quantifiers(contd.)

Proof.

Forward direction.

Assume $m' \models \forall y_1, ...y_n.\exists x.F(x)$. Therefore, $m' \models \exists x.F(x)$(why?).

Due to the last theorem, there is m such that
$m \models \exists x.F(x) \Rightarrow F(f(y_1, \dots, y_n))$.

Since m and m' only differ on f, $m \models \forall y_1, ...y_n.\exists x.F(x)$.

Therefore, $m \models \forall y_1, ..., y_n.\ F(f(y_1, .., y_n))$. $\square$

Skolemization of prenex sentence

Since the quantifiers are in prenex form, all $\exists$s can be removed using skolem functions.

Skolemization should be applied from out to inside, i.e.,

remove outermost $\exists$ first.

Example 19.3

Let us skolemize the following sentence

- $\exists z.\forall w.\exists x.\forall y.\neg(R(x,y) \Rightarrow R(w,z))$
- *Since there are no universals before $\exists z$, we introduce a function $c/0$.*
  $\forall w.\exists x.\forall y.\neg(R(x,y) \Rightarrow R(w,c))$
- *Since there is a universal $\forall w$ before $\exists x$, we introduce a function $f/1$.*
  $\forall w.\forall y.\neg(R(f(w),y) \Rightarrow R(w,c))$

Attendance quiz

Which of the following are correct skolemization of a sentence?

$\exists y \forall x.R(x, y)$ transforms to $\forall x.R(x, c)$.
$\forall x.\exists y.R(x, y)$ transforms to $\forall x.R(x, f(x))$.
$\forall x.\exists y.R(f(x), y)$ transforms to $\forall x.R(f(x), g(x))$.
$\forall x.\exists y.\forall z.R(x, y, z)$ transforms to $\forall x.\forall z.R(x, f(x), z)$.
$\forall x.\forall z.\exists y.R(x, y, z)$ transforms to $\forall x.\forall z.R(x, f(x, z), z)$.

$\exists y \forall x.R(x, y)$ transforms to $\forall x.R(x, f(x))$.
$\forall x.\exists y.R(x, y)$ transforms to $\forall x.R(x, x)$.
$\forall x.\exists y.R(f(x), y)$ transforms to $\forall x.R(f(x), f(x))$.
$\forall x.\exists y.\forall z.R(x, y, z)$ transforms to $\forall x.\forall z.R(x, f(x, z), z)$.
$\forall x.\forall z.\exists y.R(x, y, z)$ transforms to $\forall x.\forall z.R(x, f(x), z)$.

Step 4: convert body of the sentence to CNF

Consider the following skolemized prenex sentence,

$$\forall x_1, \ldots, x_n.\ F.$$

Since F is quantifier-free and build using propositional connectives over atoms, we can convert F into CNF

$$\forall x_1, \ldots, x_n.\ C_1 \land \cdots \land C_k.$$

Example 19.4

Consider sentence $\forall w.\forall y.\neg(R(f(w), y) \Rightarrow R(w, c))$*.*
After converting the body of sentence into CNF

$$\forall w.\forall y.(R(f(w), y) \land \neg R(w, c)).$$

Exercise 19.3

We may use Tseitin encoding to obtain CNF, which introduces fresh variables. What is the quantifier over the fresh propositional variables?

STEP 5: drop of explicit mention of quantifiers

Consider the following skolemized prenex clauses,

$$\forall x_1, \ldots, x_n.\ C_1 \wedge \cdots \wedge C_k.$$

Since $\forall$ distributes over $\wedge$, we translate to

$$(\forall x_1, \ldots, x_n.\ C_1) \wedge \cdots \wedge (\forall x_1, \ldots, x_n.\ C_k).$$

We may view the above sentence as conjunction of clauses

$$C_1 \wedge \cdots \wedge C_k,$$

without any explicit mention of quantifiers.
We will assume that all the free variables are universally quantified.

Example 19.5

We write sentence as $R(f(w), y) \wedge \neg R(w, c)$

Topic 19.2

Problems

Skolemization

Exercise 19.4

Demonstrate that skolemization does not produce equivalent formula.

FOL CNF

Exercise 19.5

Convert the following formula in FOL CNF

$$\exists z.(\exists x.Q(x,z) \vee \exists x.P(x)) \Rightarrow \neg(\neg\exists x.P(x) \wedge \forall x.\ \exists z.Q(z,x)\ )$$

Convert into CNF

Exercise 19.6

Consider the following formulas

$$\Sigma = \{\, \forall x, y, z.\ (z \in x \Leftrightarrow z \in y) \Rightarrow x \approx y,\\ \forall x, y.\ (x \subseteq y \Leftrightarrow \forall z.\ (z \in x \Rightarrow z \in y)),\\ \forall x, y, z.\ (z \in x - y \Leftrightarrow (z \in x \land z \notin y))\}.$$

Convert the following formula into FOL CNF.

$$\bigwedge \Sigma \land \neg \forall x, y.\ x \subseteq y \Rightarrow \exists z.(y - z \approx x)$$

Theorem prover

Exercise 19.7

Download EPROVER *a first order theorem prover from the following url.*

`http://wwwlehre.dhbw-stuttgart.de/~sschulz/E/Usage.html`

Run the prover to prove the validity of the following sentence.

$$\forall x.\exists y.\forall z.\exists w.(R(x,y) \lor \neg R(w,z))$$

Report the proof generated by the prover. Explain the proof steps.

End of Lecture 19