

# CS228 Logic for Computer Science 2021

## Lecture 2: Propositional logic - syntax and parsing

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## Topic 2.1

### Propositional logic - Syntax

# Syntax

We need a **quick method** of identifying if a group of symbols is a logical argument.

We usually define a syntax.

## Example 2.1

*Grammar of English*

Let us define syntax for propositional logic

# Propositions

The logic is over a list of propositions.

- ▶ Sky is blue
- ▶ Sun is hot
- ▶ ... *many more*

We do not care what each one says. We give each one of them a symbol.

# Propositional variables

We assume that there is a countably-infinite set  $\text{Vars}$  of propositional variables.

- ▶ Since  $\text{Vars}$  is countable, we assume that variables are **indexed**.

$$\text{Vars} = \{p_1, p_2, \dots\}$$

- ▶ The variables are just **names/symbols** without inherent meaning
- ▶ We may also use  $p, q, r, \dots, x, y, z$  to denote the propositional variables
- ▶ Propositional variables are also called **Boolean variables**

**Commentary:** All results presented in this course are extendable to uncountable  $\text{Vars}$ . For the uncountable setting, we need transfinite induction. We will ignore those extensions.

## Logic connects the variables

A logical argument connects the propositions.

Let us list all the possible ways of connecting them.

# True and false

We should be able to talk about

- ▶ *always true* statement
- ▶ *always false* statement

## Example 2.2

- ▶ *An apple is an apple*
- ▶ *I like Apple **and** I do not like Apple*

*always true*

*always false*

# Logical connectives: Not, And, and Or

We may also need ability to say

- ▶ a statement that says **negation** of another
- ▶ two statements are true at **the same time**
- ▶ **at least one of the two** statements are true

## Example 2.3

- ▶ *The apple is **not** sweet.*
- ▶ *The apple is sweet **and** Delhi is far.*
- ▶ *The apple is sweet **or** Delhi is far.*



## More logical connectives: Implies, equality, and disequality

We may also need ability to say

- ▶ Implication

if a statement is true then some other statement is also true

- ▶ Equivalence

truth value of two statements are same

- ▶ Disequality

truth value of two statements are different

- ▶ Usually called *exclusive or*, meaning exactly one of the two is true

### Example 2.4

- ▶ *If I work then I make money.*

*(implication)*

- ▶ *I like an apple if and only if I like a pen.*

*(equivalence)*

- ▶ *A is here or B is here, but both are not here.*

*(exclusive or)*

## Logical connectives

The following 10 symbols are called **logical connectives**.

formal name	symbol	read as	
true	$\top$	top	} 0-ary symbols
false	$\perp$	bot	
negation	$\neg$	not	} unary symbols
conjunction	$\wedge$	and	} binary symbols
disjunction	$\vee$	or	
implication	$\Rightarrow$	implies	
equivalence	$\Leftrightarrow$	if and only if	
exclusive or	$\oplus$	xor	} punctuation
open parenthesis	(		
close parenthesis	)		

We assume that the logical connectives are not in Vars.

# Propositional formulas

A propositional formula is a **finite string** containing symbols in Vars and logical connectives.

## Definition 2.1

*The set of propositional formulas is the smallest set  $P$  such that*

- ▶  $\top, \perp \in P$
- ▶ *if  $p \in \text{Vars}$  then  $p \in P$*
- ▶ *if  $F \in P$  then  $\neg F \in P$*
- ▶ *if  $\circ$  is a binary symbol and  $F, G \in P$  then  $(F \circ G) \in P$*

## Some notation

### Definition 2.2

$\top, \perp$ , and  $p \in \text{Vars}$  are *atomic formulas*.

### Definition 2.3

For each  $F \in \mathcal{P}$ , let  $\text{Vars}(F)$  be the set of variables appearing in  $F$ .

# Examples of propositional formulas

## Exercise 2.1

Which of the following are in P?

- ▶  $\top \Rightarrow \perp$  ✗
- ▶  $(\top \Rightarrow \perp)$  ✓
- ▶  $(p_1 \Rightarrow \neg p_2)$  ✓
- ▶  $(p_1)$  ✗
- ▶  $\neg\neg\neg\neg\neg\neg\neg\neg\neg p_1$  ✓

Not all strings over Vars and logical connectives are in P.

We need a method to recognize a string belongs to P or not.

How can we argue that a string does or does not belong to P?

## Topic 2.2

### Encoding arguments into logic

## Example : symbolic argument

### Example 2.5

We have seen the following argument.

*If  $c$  then if  $s$  then  $f$ . not  $f$ . Therefore, if  $s$  then not  $c$ .*

where

- ▶  $c$  = the seed catalogue is correct
- ▶  $s$  = seeds are planted in April
- ▶  $f$  = the flowers bloom in July

We can write the above argument as propositional formula as follows

$$\left( \left( \underbrace{(c \Rightarrow (s \Rightarrow f))}_{\text{Premise 1}} \right) \wedge \underbrace{(\neg f)}_{\text{Premise 2}} \right) \Rightarrow \underbrace{(s \Rightarrow \neg c)}_{\text{Conclusion}}$$

## Example: symbolizing bad and good puzzle

### Problem Context

#### Example 2.6

*The good people always tell the truth and the not good people always tell a lie. Now let us consider the following puzzle.*

*There are two people A and B. A says, "I am not good or B is good". What are A and B?*

*Let us give symbols to propositions:*

- ▶  $p_A = A$  is good.
- ▶  $p_B = B$  is good.

*Therefore, we encode the puzzle as follows.*

$$\underbrace{((\neg p_A \vee p_B))}_{\text{Statement of A}} \Leftrightarrow p_A$$

*To solve the puzzle, we need a **satisfying** assignment to the formula.*

**Commentary:** The puzzle is borrowed from *What is The Name of This Book?* by Raymond M. Smullyan



## Topic 2.3

### Parsing formulas

## Parse tree

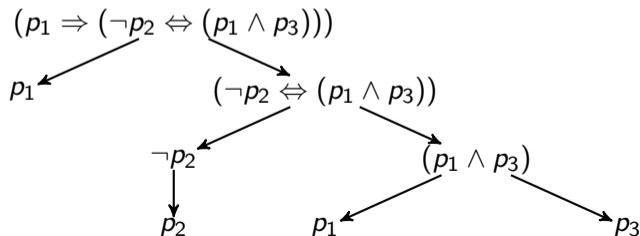
$F \in P$  iff  $F$  is obtained by unfolding of the generation rules

### Definition 2.4

A *parse tree* of a formula  $F \in P$  is a tree such that

- ▶ the root is  $F$ ,
- ▶ leaves are atomic formulas, and
- ▶ each internal node is formed by applying some formation rule on its children.

### Example 2.7



# Parse tree and unique parsing

## Theorem 2.1

$F \in P$  iff there is a parse tree of  $F$ .

### Proof.

The reverse direction is immediate. In the forward direction, we prove a stronger theorem, i.e., **existence of unique parsing tree.** □

## Theorem 2.2

Each  $F \in P$  has a unique parsing tree.

### Proof.

The proof is at the last section of the slides. □

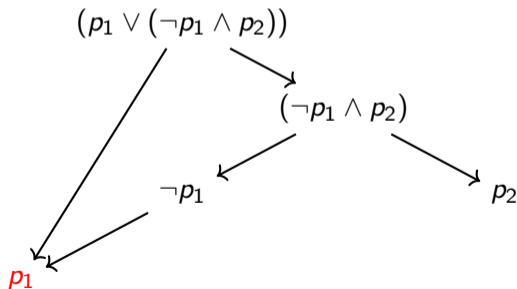
## Parse tree is a directed-acyclic graph (DAG)

We have been thinking that the parsing produces parse tree.

However, the parsing produces a parse DAG.

### Example 2.8

Consider formula  $(p_1 \vee (\neg p_1 \wedge p_2))$ . The following is the parse tree of the above formula.



# Subformula

## Definition 2.5

A formula  $G$  is a **subformula** of formula  $F$  if  $G$  occurs within  $F$ .  $G$  is a **proper subformula** of  $F$  if  $G \neq F$ . Let  $sub(F)$  denote the set of subformulas of  $F$ .

The nodes of the parse tree of  $F$  form the **set of subformulas** of  $F$ .

## Definition 2.6

**Immediate subformulas** are the children of a formula in its parse tree, and **leading connective** is the connective that is used to join the children.

## Example 2.9

$$sub((\neg p_2 \Leftrightarrow (p_1 \wedge p_3))) = \{(\neg p_2 \Leftrightarrow (p_1 \wedge p_3)), \neg p_2, (p_1 \wedge p_3), p_1, p_2, p_3\}$$

The leading connective of  $F$  is  $\Leftrightarrow$ .

**Commentary:** Note that the above definition does not allow  $p_2 \Leftrightarrow (p_1 \wedge p_3)$  to be a subformula of  $F$ , because  $p_2 \Leftrightarrow (p_1 \wedge p_3)$  is not a formula. In later discussions, we may drop parenthesis in our writings and it may cause confusion. So, when we apply the above definition we need to keep the invisible parentheses in our mind.

## Topic 2.4

### Shorthand

## Too many parentheses

In the above syntax, we need to write a large number of parentheses.

Using **precedence order over logical connectives**, we may drop some parentheses without losing the unique parsing property.

### Example 2.10

Consider  $((p \wedge q) \Rightarrow (r \vee p))$

- ▶ *We may drop outermost parenthesis without any confusion*

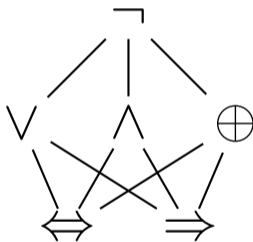
$$(p \wedge q) \Rightarrow (r \vee p)$$

- ▶ *If  $\wedge$  and  $\vee$  get precedence over  $\Rightarrow$  during parsing, we do not need the rest of parentheses*

$$p \wedge q \Rightarrow r \vee p$$

# Precedence order

We will use the following precedence order in writing the propositional formulas





## Using precedence order

Consider the following formula for  $n > 1$

$$F_0 \circ_1 F_1 \circ_2 F_2 \circ_3 \cdots \circ_n F_n,$$

where  $F_0, \dots, F_n$  are either atomic or enclosed by parentheses, or their negation.

We transform the formula as follows

- ▶ Find an  $\circ_i$  such that  $\circ_{i-1}$  and  $\circ_{i+1}$  have lower precedence if they exist.
- ▶ Introduce parentheses around  $F_{i-1} \circ_i F_i$  and call it  $F'_i \triangleq (F_{i-1} \circ_i F_i)$ .

$$F_0 \circ_1 \cdots \circ_{i-1} F_{i-2} \circ_{i-1} F'_i \circ_{i+1} F_{i+1} \circ_{i+2} \cdots \circ_n F_n$$

We apply the above until  $n = 1$  and then apply the normal parsing.

Inside of  $F'_i$ s may also have ambiguities, which are recursively resolved using the above procedure.

**Commentary:** We have not presented the above as a formal algorithm. However, we can present the procedure in the above style. You will find many computer science texts do not write their algorithms in a formal presentation to avoid cumbersome notation. Please learn to handle.

## Example: parsing using the precedence order

### Example 2.11

Consider formula  $p \wedge q \Rightarrow r \vee p$ . Let us try to bring back the parentheses.

$\Rightarrow$  has lower precedence than  $\wedge$ , therefore we can group neighbours of  $\wedge$

$$(p \wedge q) \Rightarrow r \vee p$$

Since  $\vee$  has higher precedence over  $\Rightarrow$ , we first group  $\vee$ .

$$(p \wedge q) \Rightarrow (r \vee p)$$

Now we can group  $\Rightarrow$  without any confusion

$$((p \wedge q) \Rightarrow (r \vee p))$$

## Example precedence order

### Example 2.12

*Which of the following formulas can be unambiguously parsed?*

- ▶  $\neg p \vee (p \oplus q) \Leftrightarrow p \wedge q$  ✓
- ▶  $p \vee q \wedge r$  ✗
- ▶  $p \vee q \vee r$  ✗
- ▶  $p \Rightarrow q \Rightarrow r$  ✗

Associativity preference may further reduce the need of parenthesis

## Associative

Problem: If a binary operator repeats, we **do not know** how to group.

Solution: we give preference to one side or another.

Let us make all our operators “**right associative**”, i.e., first group the rightmost occurrence.

### Example 2.13

Consider formula  $p \Rightarrow q \Rightarrow r$ .

We first group the right  $\Rightarrow$ :  $p \Rightarrow (q \Rightarrow r)$

Then, we group the left  $\Rightarrow$ :  $(p \Rightarrow (q \Rightarrow r))$

**Commentary:** Not all operators are affected by associativity. For example,  $\wedge$  and  $\vee$  operators have same meaning if we use any order of associativity. On the other hand,  $\Rightarrow$  needs a convention for associativity.

### Exercise 2.2

Modify the parsing procedure of the earlier slide to support the above.

# Substitution

## Definition 2.7

For  $F \in \mathcal{P}$  and  $p_1, \dots, p_k \in \text{Vars}$ , let  $F[G_1/p_1, \dots, G_k/p_k]$  denote another formula obtained by *simultaneously* replacing all occurrence of  $p_i$  by a formula  $G_i$  for each  $i \in 1..k$ .

## Example 2.14

1.  $(p \Rightarrow (r \Rightarrow p))[(r \oplus s)/p] = ((r \oplus s) \Rightarrow (r \Rightarrow (r \oplus s)))$
2.  $(p \Rightarrow (r \Rightarrow p))[(r \oplus s)/p, x/r] \neq (p \Rightarrow (r \Rightarrow p))[(r \oplus s)/p][x/r]$

## Exercise 2.3

- a. Definition 2.7 is informal. Give a formal definition.
- b. Write the result of substitutions in the second example.
- c. Give a most general restriction on substitutions such that simultaneous and sequential substitutions produce the same result.

## Notation for substitution

For shorthand, we may write a formula  $F$  as

$$F(p_1, \dots, p_k),$$

where we say that variables  $p_1, \dots, p_k$  play a special role in  $F$ .

$$\text{Let } F(G_1, \dots, G_n) \text{ be } F[G_1/p_1, \dots, G_k/p_k].$$

### Example 2.15

$$\text{Let } F(p, q) = \neg p \oplus q$$

$$F(r \vee q, \top) = \neg(r \vee q) \oplus \top$$

## Topic 2.5

### Problems

## Exercise: symbolizing bad and good puzzle\*\*

### Exercise 2.4

*People are either good or bad. The good people always tell the truth and the bad people always tell a lie. Now let us consider the following puzzle.*

*There are two people A and B. A said some thing, but we could not hear. B said, "A is saying that A is bad". What are A and B?*

*Encode the above puzzle into a propositional logic formula.*

**Commentary:** The puzzle is borrowed from *What is The Name of This Book?* by Raymond M. Smullyan. It is difficult to encode because we also need to model internal mental state of the participants.



## Exercise: more puzzles

### Exercise 2.5

*The following facts are known about three suspects X, Y, and Z of a crime.*

- 1. If X is guilty and Y is innocent, then Z is guilty.*
- 2. If Z is guilty, then one of the other two worked with Z.*
- 3. X never works with Z.*
- 4. At least one of them is guilty.*

*Is X necessarily guilty?*

*Encode the puzzle into a formula, whose satisfiability gives the answer of the puzzle? Please note that the purpose of the question is the encoding not solving the puzzle.*

## Exercise: more puzzles

### Exercise 2.6

*People are either good or bad. The good people always tell the truth and the bad people always tell a lie. Now let us consider the following puzzle.*

*There are three people A, B, and C. A said, "All of us are bad.". B said, "Exactly one of us is good.". What are A, B, and C?*

*Encode the above puzzle into a propositional logic formula.*

## Let expression

We may extend the grammar of propositional logic with let expressions.

$$(\text{let } p = F \text{ in } F)$$

Let-expression is a syntactic device to represent large formulas succinctly.

$$(\text{let } p = F \text{ in } G) \text{ represents } G[F/p]$$

### Example 2.16

$$(\text{let } p = (q \wedge r) \text{ in } ((p \wedge s) \vee (q \Rightarrow \neg p))) \quad \text{represents} \quad ((q \wedge r) \wedge s) \vee (q \Rightarrow \neg(q \wedge r))$$

### Exercise 2.7

*Give an example in which let expressions can represent a formula in exponentially less space.*

# Precedence order

## Exercise 2.8

*Add minimum parentheses in the following formulas such that they have unique parsing under our precedence order*

1.  $p \wedge q \vee r \wedge s \wedge t \vee u \vee v \wedge w$

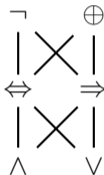
2.  $p \Rightarrow \neg q \oplus p \vee p \wedge \neg r \Leftrightarrow s \wedge t$

**Commentary:** Please work the above problems with and without associative preference rules. In the exams, we will make it clear.

# Custom precedence order

## Exercise 2.9

Consider the following precedence order



Add minimal parentheses in the following formulas such that they have unique parsing tree

1.  $\neg p \Rightarrow q \wedge r \Rightarrow p \Rightarrow q$
2.  $p \Rightarrow \neg q \oplus p \vee p \wedge \neg r \Leftrightarrow s \wedge t$

**Commentary:** Please work the above problems with and without associative preference rules. In the exams, we will make it clear.

## Topic 2.6

Extra lecture slides: unique parsing

# Matching parentheses

## Theorem 2.3

*Every  $F \in P$  has matching parentheses, i.e., equal number of '(' and ')'.*

Proof.

**base case:**

atomic formulas have no parenthesis. Therefore, matching parenthesis

**induction steps:**

We assume  $F, G \in P$  has matching parentheses.

Let  $n_F$  and  $n_G$  be the number of '(' in  $F$  and  $G$  respectively.

Trivially,  $\neg F$  has matching parentheses.

For some binary symbol  $\circ$ , the number of both '(' and ')' in  $(F \circ G)$  is  $n_F + n_G + 1$ .

Due to the structural induction, the property holds. □

## Prefix of a formula

### Theorem 2.4

*A proper prefix of a formula is not a formula.*

### Proof.

We show a proper prefix of a formula is in one of the following forms.

1. strictly more '(' than ')',
2. a (possibly empty) sequence of  $\neg$ .

Clearly, both the cases **are not** in P.

### base case:

A proper prefix of atomic formulas is empty string, which is the second case

...

### Exercise 2.10

*Give examples of the above two cases*



## Prefix of a formula II

Proof(contd.)

**induction step:**

Let  $F, G \in P$ .

Consider proper prefix  $F'$  of  $\neg F$ . There are two cases.

- ▶  $F' = \epsilon$ , case 2
- ▶  $F' = \neg F''$ , where  $F''$  is a proper prefix of  $F$ . Now we again have two subcases for  $F''$ .
  - ▶ If  $F''$  is in case 1,  $F'$  belongs to case 1
  - ▶ If  $F'' = \neg.. \neg$ ,  $F'$  belongs to case 2

...

## Prefix of a formula III

### Proof(contd.)

By induction  $F$  and  $G$  have balanced parenthesis.

Consider proper prefix  $H$  of  $(F \circ G)$ ,  $F'$  be prefix of  $F$ , and  $G'$  be prefix of  $G$ .

- ▶ If  $H = (F \circ G$ ,  $H$  belongs to case 1 because  $H$  has one extra '('
- ▶ If  $H = (F \circ G'$ ,  $H$  belongs to case 1<sub>(why?)</sub>

Similarly the following cases are handled

- ▶  $H = (F \circ$
- ▶  $H = (F'$
- ▶  $H = ($

□

### Exercise 2.11

Complete the *(why?)*.

# Unique parsing

## Theorem 2.5

Each  $F \in P$  has a unique parsing tree.

**Proof.**

$\nu(F) \triangleq$  number of logical connectives in  $F$ . We apply induction over  $\nu(F)$ .

**base case:**  $\nu(F) = 0$

$F$  is an atomic formula, therefore has a single node parsing tree.

**inductive steps:**  $\nu(F) = n$

We assume that each  $F'$  with  $\nu(F') < n$  has a unique parsing tree.

*case*  $F = \neg G$ : Since  $G$  has a unique parsing tree,  $F$  has a unique parsing tree.

*case*  $F = (G \circ H)$ :

Suppose there is another formation rule such that  $F = (G' \circ' H')$ .

Since  $F = (G \circ H) = (G' \circ' H')$ ,  $G \circ H = G' \circ' H'$ .

Without loss of generality,  $G$  is prefix of  $G'$ .

Since  $G, G' \in P$ ,  $G$  can not be proper prefix of  $G'$ . Therefore,  $G = G'$ .

Therefore,  $\circ = \circ'$ . Therefore,  $H = H'$ . Therefore, only one way to unfold  $F$ .

$F$  has a unique parsing tree. □

# Parsing algorithm

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## Algorithm 2.1: PARSER

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**Input:**  $F$  : a string over Vars and logical connectives

**Output:** parse tree if  $F \in P$ , exception FAIL otherwise

**if**  $F = p$  or  $F = \top$  or  $F = \perp$  **then return**  $(\{F\}, \emptyset)$  ;

**if**  $F = \neg G$  **then**

$(V, E) := \text{PARSER}(G)$ ;

**return**  $(V \cup \{F\}, E \cup \{(F, G)\})$ ;

**if**  $F$  has matching parentheses and  $F = (F')$  **then**

$G :=$  smallest prefix of  $F'$  where non-zero parentheses match or atom after a sequence of ' $\neg$ 's;

$o'H := \text{tail}(F', \text{len}(G))$ ;

**if** the above two match succeed **then**

$(V_1, E_1) := \text{PARSER}(G)$ ;

$(V_2, E_2) := \text{PARSER}(H)$ ;

**return**  $(V_1 \cup V_2 \cup \{F\}, E_1 \cup E_2 \cup \{(F, G), (F, H)\})$ ;

**Throw** FAIL

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# Parse Algorithm

## Exercise 2.12

Show the run of Algorithm 2.1 on the following formulas.

1.  $\neg q \Rightarrow (p \oplus r \Leftrightarrow s)$
2.  $(\neg(p \Rightarrow q) \wedge (r \Rightarrow (p \Rightarrow q)))$

End of Lecture 2