CS 433 Automated Reasoning 2021

Lecture 15: Other methods for LRA

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Ideas that do not work!!

We have been seeing the success of SMT solvers and their algorithms.

Today, we will look at the methods that may (or may not) look efficient on paper but do not perform well!

This lecture is a word of caution, we may have stuck in a local maxima of ideas!!

Where are going to look a few less popular methods

- Fourier-Motzkin
- Interior point methods
- Ellipsoid method
- Kermaker's method

Topic 15.1

Fourier-Motzkin

Fourier-Motzkin

The algorithm proceeds by eliminating variables one by one. After eliminating all the variables, if the input reduces to $\top$ then only the input is satisfiable.

For variable x, a conjunction of linear inequalities can be transformed into the following form, where x does not occur in the linear terms s_j, t_k, and u_k.

$$\bigwedge_{j=1}^{m} s_j \leq x \wedge \bigwedge_{i=1}^{\ell} x \leq t_i \wedge \bigwedge_{k=1}^{n} u_k \leq 0$$

$$\Updownarrow$$

$$\bigwedge_{j=1}^{m} \bigwedge_{i=1}^{\ell} s_j \leq t_i \wedge \bigwedge_{k=1}^{n} u_k \leq 0$$

equisatisfiable but without x

Exercise 15.1

a. Add support for equality, dis-equality, and strict inequalities

b. What is the complexity?

Example: Fourier-Motzkin

Example 15.1 *Consider:* $-x_1 + x_2 + 2x_3 \leq 0 \land x_1 - x_2 \leq 0 \land x_1 - x_3 \leq 0 \land 1 - x_3 \leq 0$

Suppose we eliminate x_1 first. We transform the constraints into our format.

$$\underbrace{x_2 + 2x_3 \leq x_1}_{x_1 \text{ lower bounded}} \land \underbrace{(x_1 \leq x_2 \land x_1 \leq x_3)}_{x_1 \text{ upper bounded}} \land \underbrace{-x_3 + 1 \leq 0}_{x_1 \text{ does not occur}}$$

Eliminated constraints: $x_2 + 2x_3 \leq x_2 \land x_2 + 2x_3 \leq x_3 \land -x_3 + 1 \leq 0$

After simplification: $x_3 \leq 0 \land x_2 + x_3 \leq 0 \land -x_3 + 1 \leq 0$

Since x_2 has no lower bound, we can drop x_2 atoms: $x_3 \leq 0 \land -x_3 + 1 \leq 0$

Eliminating x_3: $1 \leq 0 \leftarrow$ *false formula therefore unsat*

Exercise 15.2

How to generate the proof for unsatisfiable conjunctions

Another example

Example 15.2

Consider: $-x_1 + x_2 + 2x_3 \leq 0 \land x_1 - x_2 \leq 0 \land x_1 - x_3 \leq 0 \land -x_1 \leq 2$

Let us transform to eliminate x_1. $\underbrace{x_2 + 2x_3 \leq x_1 \land -2 \leq x_1}_{x_1 \text{ lower bounded}} \land \underbrace{(x_1 \leq x_2 \land x_1 \leq x_3)}_{x_1 \text{ upper bounded}}$

Eliminated constraints: $x_2 + 2x_3 \leq x_2 \land x_2 + 2x_3 \leq x_3 \land -2 \leq x_2 \land -2 \leq x_3$

After simplification: $x_3 \leq 0 \land x_2 + x_3 \leq 0 \land -2 \leq x_2 \land -2 \leq x_3$

Eliminating x_2*:* $x_3 \leq 0 \land -2 \leq -x_3 \land -2 \leq x_3$

After simplification: $x_3 \leq 0 \land -2 \leq x_3$

Eliminating x_3*:* $-2 \leq 0 \leftarrow$ *true formula therefore sat*

Exercise 15.3

How to generate the model for satisfiable conjunctions?

Fourier-Motzkin in practice

Both complexity and practical performance of the algorithm are bad.

Almost never used in practice, except for some bounded simplifications.

Topic 15.2

Ellipsoid method

Ellipsoid method : a Soviet scare

Khachian found the first polynomial time method for linear programming.

ARCHIVES | 1979

A Soviet Discovery Rocks World of Mathematics

By MALCOLM W. BROWNE NOV. 7, 1979


The New York Times

Similar Transportation Bond Issues Are Passed in New York and Jersey

CARTER SAID TO ASK FOR LOYALTY CHECK OF AIDES ON POLITICS

IRAN'S CIVIL GOVERNMENT OUT; HOSTAGES FACE DEATH THREAT; OIL EXPORTS BELIEVED HALTED

STUDENTS WARN U.S.

A surprise discovery by an obscure Soviet mathematician has rocked the world of mathematics and computer analysis, and experts have begun exploring its practical applications.

http://www.nytimes.com/1979/11/07/archives/a-soviet-discovery-rocks-world-of-mathematics-russians-surprise.html

Elliposid method

We want to check satisfiability of

$$Ax \leq b.$$

We assume that space $Ax \leq b$ is bounded and full dimensional.

Sizes of numbers

Size are define as follows

- For an integer r, $size(r) \triangleq \log_2(r)$
- For a rational number p/q, $size(p/q) \triangleq size(p) + size(q)$
- For a row/vector c, $c \triangleq n + \sum_i size(c_i)$
- For a matrix A, $size(A) \triangleq mn + \sum_{ij} size(A_{ij})$

Coeffcient bounded solution existence

Theorem 15.1

If solution exists of $Ax \leq b$, then there is a solution with size less than $4n^2\phi$, where ϕ is the maximum row size.

Proof.

Every vertex is a solution of n equalities $A'x = b'$ from $Ax \leq b$.

The vertex is $x = A'^{-1}b'$, which depends on the size of determinant of A'.

Since determinant sums multiples of n numbers, its size is bounded by $2n\phi$.

Therefore, $size(x_i) \leq 4n\phi$. Therefore, $size(x) \leq 4n^2\phi$. □

Minimum volume condition

Theorem 15.2

If $Ax \leq b$ is satisfiable, volume of $Ax \leq b$ is bigger than $2^{-2n(4n^2\phi)}$

Proof sketch.

Since $Ax \leq b$ is sat, bounded, and full-dimensional, there are $n+1$ affinely independent vertices $x_0, ..., x_n$ of $Ax \leq b$.

Since A and b have bounded precision, the size of rational numbers in $x_0, ..., x_n$ are bounded.

Therefore, the non-zero volume of simplex $x_0, ..., x_n$ is $\frac{|det([x_1 - x_0 \;\; x_2 - x_0 \;\; \dots \;\; x_n - x_0])|}{n!}$.

Therefore, the denominator of the volume is bounded. Since the simplex is contained inside $Ax \leq b$, there is a lower bound on the volume. □

Commentary: affinely independent vertices means that not all $n+1$ points are in some $n-1$ dimensional affine space. We have skipped the exact calculation on the lower bound. Not too difficult to do it yourself.

Idea!

- We know that if there is a solution, it is in the finite space.
- The finite space can be divided in finite granularity!!
- If we can iteratively divide the space, we may have an efficient algorithm.

We use ellipses to describe and split the finite space.

Topic 15.3

Understanding ellipses

Positive definite matrix

Definition 15.1

A symmetric matrix is positive definite if all its eigenvalues are positive.

Theorem 15.3

The following statements are equivalent

1. *D is positive definite*
2. *$D = B^T B$ for some non-singular B*
3. *For each $x \neq 0$, $x^T D x > 0$*

Proof.

(1)⇒(2)

- D can be diagonalized, i.e., $D = P^T D' P$ where D' is a diagonal matrix
- Since all eigenvalues are positive we can split $D' = D'' D''$. Therefore, $B = P^T D''$ □

Exercise 15.4

a. Prove (2)⇒(3) *b. Prove (3)⇒(1)*

Represnting ellipses

Definition 15.2

The following defines interior of a ellipse

$$ell(z, D) := \{x | (x - z)^T D^{-1} (x - z) \leq 1\}$$

where is D a $n \times n$ positive definite matrix.

- z is the center of the ellipse
- D defines the direction and length of axes

Example 15.3

2-D unit ball is

$$ell(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathtt{I}) = \{x | x^T \mathtt{I} x \leq 1\} = \{x | \ |x|^2 \leq 1\}$$

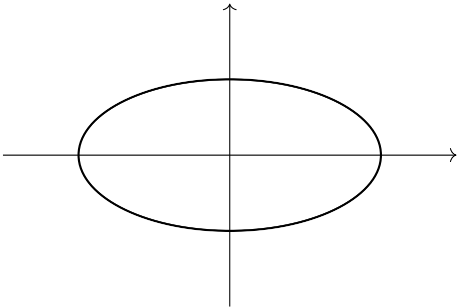
Shorthand for ball $ball(z, r) := ell(z, r^2 \mathtt{I})$

Ellipse example: stretched

Example 15.4

Ellipse $x_1^2 + 4x_2^2 \leq 4$ *will be encoded as follows*

$$ell(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}4 & 0\\0 & 1\end{bmatrix}) = \{x | x^T \begin{bmatrix}1/4 & 0\\0 & 1\end{bmatrix} x \leq 1\}$$

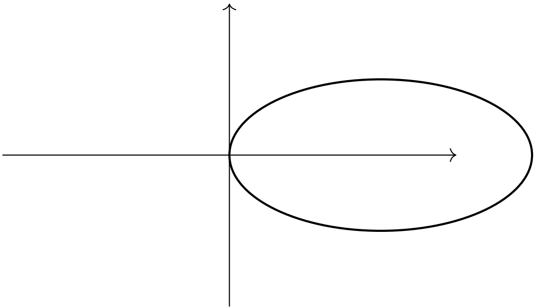


Ellipse example: shifted

Example 15.5

Ellipse $x_1^2 - 4x_1 + 4 + 4x_2^2 \leq 4$ *will be encoded as follows*

$$ell(\begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}4 & 0\\0 & 1\end{bmatrix}) = \{x | (x - \begin{bmatrix}2\\0\end{bmatrix})^T \begin{bmatrix}1/4 & 0\\0 & 1\end{bmatrix}(x - \begin{bmatrix}2\\0\end{bmatrix}) \leq 1\}$$

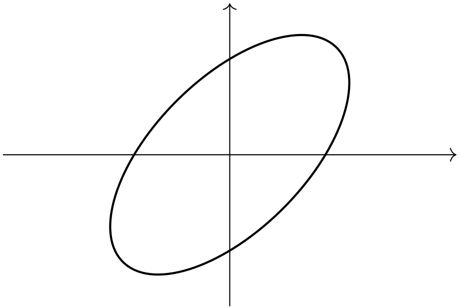


Ellipse example: rotated

Example 15.6

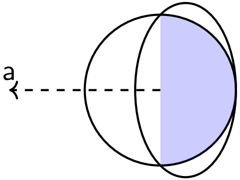
Ellipse $5x_1^2 + 5x_2^2 - 6x_1x_2 \leq 8$ *will be encoded as follows*

$$ell([0,0], \begin{bmatrix} \frac{5}{8} & \frac{-3}{8} \\ \frac{-3}{8} & \frac{5}{8} \end{bmatrix}^{-1}) = \{x | x^T \begin{bmatrix} \frac{5}{8} & \frac{-3}{8} \\ \frac{-3}{8} & \frac{5}{8} \end{bmatrix} x \leq 1\}$$



Smallest covering ellipse

What is the smallest ellipse in area that covers a half circle?



Let a be the unit row vector that defines the half circle, i.e., $Ball(0,1) \cap ax \leq 0$.

The smallest ellipse is

$$ell\left(-\frac{a^T}{n+1}, \frac{n^2}{n^2-1}\left(\mathrm{I} - \frac{2a^Ta}{n+1}\right) \right)$$

Volume decays exponentially

One axis of the ball is shrunk by $\frac{n}{n+1}$.

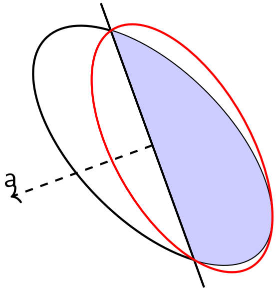
And the other $n-1$ axes are expanded by $\left(\frac{n^2}{n^2-1}\right)^{1/2}$.(why?)

Therefore the volume is changed by the factor of $\frac{n}{n+1}\left(\frac{n^2}{n^2-1}\right)^{(n-1)/2}$

$$\frac{n}{n+1}\left(\frac{n^2}{n^2-1}\right)^{(n-1)/2} < e^{-1/(2n+2)}.$$ (why?)

In more general form

Consider the following ellipse $ell(z, D)$ and direction a



The smaller $\textsc{CoveringEllipse}(z, D, a) :=$

$$ell\left(z - \frac{Da^T}{(n+1)\sqrt{aDa^T}} , \frac{n^2}{n^2-1}\left(D - \frac{2Da^TaD}{(n+1)\sqrt{aDa^T}}\right) \right)$$

Since all ellipse are linear transformations of unit ball, the exponential volume reduction still holds.

Topic 15.4

Ellipsoid method (algorithm)

Ellipsoid method

Input $Ax \leq b$.

We know if $(Ax \leq b)$ is sat, $(Ax \leq b) \subseteq ball(0, 2^{4n^2\phi})$.

Let us suppose we have initial ellipse $ell(z, D) := ball(0, 2^{4n^2\phi})$.

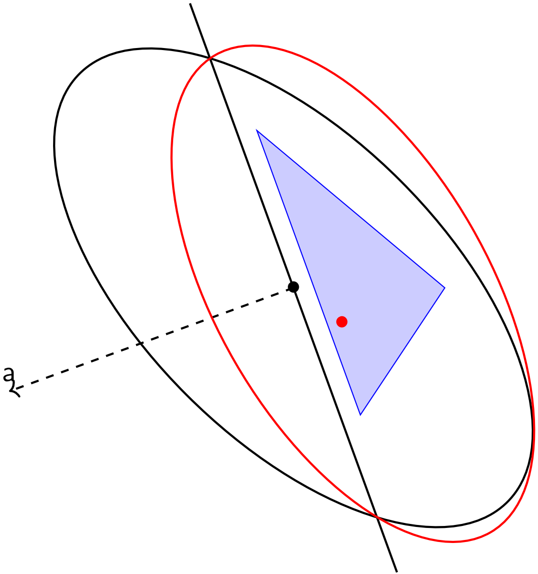
1. if z satisfies $Ax \leq b$, **return** z
2. Otherwise, find inequality $ax \leq \delta$ in $Ax \leq b$ such that $az > \delta$
3. $z, D := \textsc{CoveringEllipse}(z, D, a)$
4. If volume of $ell(z, D)$ is too small, **return** unsatisfiable
5. goto 1

Exercise 15.5

a. Why $ax \leq \delta$ exists at 2?

b. Why smaller ellipses will continue to contain $Ax \leq b$?

Ellipsoid method illustration



Ellipsoid method is polynomial

Theorem 15.4

Ellipsoid method runs less than $16n^2(4n^2\phi)$ *iterations.*

Proof.

1. Initial ellipse has volume less that $(2 \times 2^{4n^2\phi})^n$
2. Volume threshold is $2^{-2n(4n^2\phi)}$
3. Ellipse sizes decrease by the factor of $e^{-1/2(n+1)}$

□

Exercise 15.6

Prove that within the $16n^2(4n^2\phi)$ *iterations the ellipse volume will reduce below the threshold*

Ineffcient method

- Number of iterations depends on the size of the numbers
- Square root needs to be computed, i.e., high precision computation (can be avoided!)
- Experiments show that it can not compete with simplex.

Topic 15.5

Karmakar's method

Karmakar's method: west strikes back

Karmakar fixed some of the problems in ellipsoid method

ARCHIVES | 1984

BREAKTHROUGH IN PROBLEM SOLVING

By JAMES GLEICK


The New York Times
AMERICANS IN POLL VIEW GOVERNMENT MORE CONFIDENTLY
TREASURY MAY ASK INCREASES IN TAXES OF SOME CONCERNS
Albany Leaders Predicting a Cut In Income Taxes
Cocaine Traffickers Kill 17 in Peru Raid On Antidrug Team
Vote Comes to a 'Homeland,' But African Problems Linger
Breakthrough in Problem Solving
Homeless Spend Nights in City Welfare Office
High Cuban Defector Speaks Out, Denouncing Castro as 'Impulsive'

A 28-year-old mathematician at A.T.&T. Bell Laboratories has made a startling theoretical breakthrough in the solving of systems of equations that often grow too vast and complex for the most powerful computers.

The discovery, which is to be formally published next month, is already circulating rapidly through the mathematical world. It has also set off a deluge of inquiries from brokerage houses, oil companies and airlines, industries with millions of dollars at stake in problems known as linear programming.

`http://www.nytimes.com/1984/11/19/us/breakthrough-in-problem-solving.html`

Efficacy of Karmakar's method

- There are claims that it is far more efficient
- No large scale study to demonstrate (as far as I know!!)
- Later further improvements on Karmakar's method were found
- However, all SMT solvers still implement simplex

Commentary: For historical context: Who Invented the Interior-Point Method? -David Shanno
https://www.math.uni-bielefeld.de/documenta/vol-ismp/20_shanno-david.pdf

Topic 15.6

Problems

Attendance quiz

Exercise 15.7

Which of the following true in Fourier-Motzkin elimination?

After eliminating x_1 *in* $x_1 + x_2 \leq 3 \wedge -x_1 + x_2 \leq 3$ *we obtain* $x_2 - 3 \leq 3 - x_2$.

After eliminating x_2 *in* $x_1 + x_2 \leq 3 \wedge -x_1 + x_2 \leq 3$ *we obtain empty constraints.*

After eliminating x_1 *in* $x_1 + x_2 \leq 3 \wedge x_1 + x_2 \leq 3$ *we obtain empty constraints.*

After eliminating x_1 *in* $2x_1 + x_2 \leq 4 \wedge -x_1 + x_2 \leq 3$ *we obtain* $x_2 - 3 \leq 2 - 0.5x_2$.

After eliminating x_2 *in* $2x_1 + x_2 \leq 4 \wedge -x_1 - x_2 \leq 3$ *we obtain* $-x_1 - 3 \leq 4 - 2x_1$.

After eliminating x_1 *in* $x_1 + x_2 \leq 3 \wedge -x_1 + x_2 \leq -3$ *we obtain* $x_2 - 3 \leq 3 - x_2$.

After eliminating x_2 *in* $x_1 - x_2 \leq 3 \wedge -x_1 + x_2 \leq 3$ *we obtain empty constraints.*

After eliminating x_1 *in* $-x_1 + x_2 \leq 2 \wedge x_1 + x_2 \leq 3$ *we obtain empty constraints.*

After eliminating x_1 *in* $2x_1 + x_2 \leq 2 \wedge -x_1 + x_2 \leq 3$ *we obtain* $x_2 - 3 \leq 2 - 0.5x_2$.

After eliminating x_2 *in* $2x_1 + x_2 \leq 4 \wedge -x_1 + x_2 \leq 3$ *we obtain* $-x_1 - 3 \leq 4 - 2x_1$.

Attendance quiz

Exercise 15.8

Which of the following true about ellipses? O is origin point.
The volume of $ell(O, diag(4,1))$ *is* 2π.
The volume of $ell(O, diag(4,4))$ *is* 4π.
The volume of $ell(O, diag(1,1))$ *is* $\sqrt{3}\pi$.
The volume of $ell(O, diag(4,3))$ *is* $4\sqrt{3}\pi$.

End of Lecture 15