CS228 Logic for Computer Science 2021

Lecture 20: Logical theories

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What is theory?

Theory

"Theory is a contemplative and rational type abstract ... thinking, or .... ."

- Wikipedia

Example 20.1

Scientific and economic theories

- *Newton's theory of gravity*
- *Theory of evolution*
- *Theory of marginal utility*

Conspiracy theories

Example 20.2

- *9/11 is an inside job*
- *Trump is a Russian mole*
- $R + L = J$ *(confirmed)*

They may sound silly.

However, they are still theories.

FOL has no knowledge

First-order logic(FOL) provides a grammar for rational abstract thinking.

However, FOL carries no knowledge of any subject matter.

It was not obvious. 16th century philosopher René Descartes tried to prove

$$\text{Inherent model of logic} \Rightarrow \text{God exists.}$$

Theory crafting needs something more than logic

$$\text{Theory} = \text{Subject knowledge} + \text{FOL}$$

Now we will formally define theories in logic.

Defining theories

The subject knowledge can be expressed in the following two ways

1. the set of acceptable models
2. the set of valid sentences in the subject

Example 20.3

Model m with $D_m = \mathbb{N}$ is the only model we consider for the theory of natural numbers.

We can also define the theory using the set of valid sentences over natural numbers. e.g. $\forall x.\ x + 1 \neq 0$.

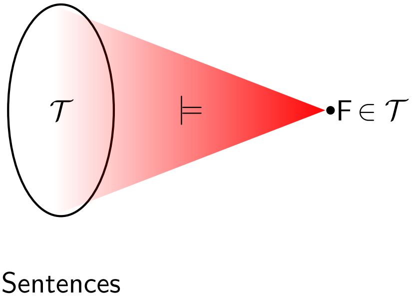
Now let us define this formally.

Theories

Definition 20.1
A *theory* $\mathcal{T}$ *is a set of sentences closed under implication, i.e.,*

$$\text{if } \mathcal{T} \models F \text{ then } F \in \mathcal{T}.$$

Abuse of notation, $\models$ is also used for implication



$\mathcal{T}$-satisfiability and $\mathcal{T}$-validity

Definition 20.2

A formula F is $\mathcal{T}$-satisfiable if there is model m such that $m \models \mathcal{T} \cup \{F\}$. We write $\mathcal{T}$-satisfiability as $m \models_{\mathcal{T}} F$.

Definition 20.3

A formula F is $\mathcal{T}$-valid if $\mathcal{T} \models F$. We write $\mathcal{T}$-validity as $\models_{\mathcal{T}} F$.

Example 20.4

Let $\mathcal{T}_{\mathbb{N}}$ be the set of true sentences in arithmetic over natural numbers.

$\exists x.\ x > 0$ is not a valid sentence in FOL. However, $\mathcal{T}_{\mathbb{N}} \models \exists x.\ x > 0$. Therefore, $\models_{\mathcal{T}_{\mathbb{N}}} \exists x.\ x > 0$.

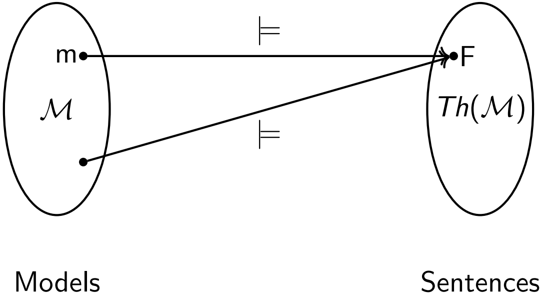
$\exists x.\ x < 0$ is a satisfiable sentence in FOL. However, $\not\models \mathcal{T}_{\mathbb{N}} \cup \{\exists x.\ x < 0\}$. Therefore, $\not\models_{\mathcal{T}_{\mathbb{N}}} \exists x.\ x < 0$.

Theory of Models

Definition 20.4

*For a set $\mathcal{M}$ of models for signature **S**, let $Th(\mathcal{M})$ be the set of **S**-sentences that are true in every model in $\mathcal{M}$, i.e.,*

$$Th(\mathcal{M}) = \{F \mid \text{for all } m \in \mathcal{M}.\ m \models F\}$$



Theory and models

Theorem 20.1
$Th(\mathcal{M})$ is a theory

Proof.
Consider F such that $Th(\mathcal{M}) \models F$.

Therefore, F is true in every model in $\mathcal{M}$.

Therefore, $F \in Th(\mathcal{M})$.

$Th(\mathcal{M})$ is closed under implication. □

Topic 20.1

Axioms

Handle on theories

Being a closed set is not a useful definition.

The definition via a set of models is also not convenient, since we use formulas in our calculations.

How do we handle a theory? Can we do better for deciding $\mathcal{T}$-validity/satisfiability?

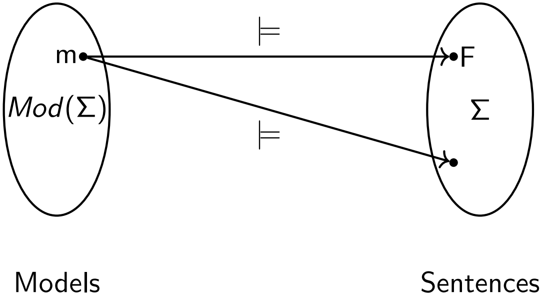
Define theories using axioms

Definability of a class of models

Definition 20.5

For a set Σ of sentences in signature **S**, *let $Mod(\Sigma)$ be a class of models such that*

$$Mod(\Sigma) = \{m \mid \text{for all } F \in \Sigma.\ m \models F\}.$$

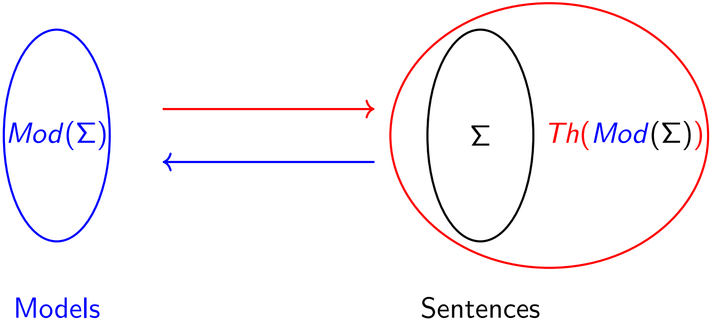


Consequences

Definition 20.6

For a set Σ of sentences, let $Cn(\Sigma)$ be the set of consequences of Σ, i.e.,

$$Cn(\Sigma) = Th(Mod(\Sigma)).$$



Exercise 20.1

Show for a theory $\mathcal{T}$, $\mathcal{T} = Cn(\mathcal{T})$.

Example: theory of lists

Example 20.5

Let us suppose our subject of interest is lists. First we need to fix our signature.

We should be interested in the following functions and predicates

- :: - *constructor for extending a list*
- *head - function to pick head of a list*
- *tail - function to pick tail of a list*
- *atom - predicate that checks if something is constructed using* :: *or not*

The signature is

$$\mathbf{S} = (\{:: /2, head/1, tail/1\}, \{atom/1\})$$

Example: theory of lists

Let Σ consists of

1. $\forall x, y.\ head(x :: y) = x$
2. $\forall x, y.\ tail(x :: y) = y$
3. $\forall x.\ atom(x) \lor head(x) :: tail(x) = x$
4. $\forall x, y.\ \neg atom(x :: y)$

$\mathcal{T}_{list} = Th(Mod(\Sigma))$ is the set of valid sentences over lists.

The sentences in $\mathcal{T}_{list}$ may not be true on the non-list models.

Exercise 20.2

Can we derive that empty list exists from the above axioms?

Axiomatizable

A set is decidable if there is an algorithm to check membership

Definition 20.7

A theory $\mathcal{T}$ is axiomatizable if there is a decidable set Σ such that $\mathcal{T} = Cn(\Sigma)$.

Definition 20.8

A theory $\mathcal{T}$ is finitely axiomatizable if there is a finite set Σ such that $\mathcal{T} = Cn(\Sigma)$.

Exercise 20.3

Show that if $Cn(\Sigma)$ is finitely axiomatizable, there is a finite $\Sigma' \subseteq \Sigma$ such that $Cn(\Sigma') = Cn(\Sigma)$.

Commentary: Solution of the above exercise: Let Σ'' be a finite axiomatization of $Cn(\Sigma)$. Therefore, $\Sigma \models \Sigma''$. Due to the compactness of FOL, there is a finite $\Sigma' \subseteq \Sigma$ such that $\Sigma' \models \Sigma''$. Therefore, $Cn(\Sigma'') \subseteq Cn(\Sigma') \subseteq Cn(\Sigma)$. Therefore, $Cn(\Sigma') = Cn(\Sigma)$.

Axiomatizable implies enumerable

We assume that theories consists of countably many symbols.

Theorem 20.2

An axiomatizable theory $\mathcal{T}$ is effectively enumerable.

Proof.

Let decidable set Σ' be such that $Cn(\Sigma') = \mathcal{T}$.
Therefore for each $F \in \mathcal{T}$, there is a finite $\Sigma_0 \subseteq \Sigma'$ such that $\Sigma_0 \models F$.(why?)

We enumerate triples (F, Σ_0, Pr), where

1. $F \in$ **S**-sentences,
2. finite $\Sigma_0 \subseteq \Sigma'$, and
3. Pr is a FO-proof (sequence of formulas with consequence relation).

Since the three sets are effectively enumerable, the set of triples is also effectively enumerable.
In the enumeration of triples, if Pr is proof of $\Sigma_0 \models F$, we report F. □

Commentary: In the above, method we can not ever say that a formula is not in $\mathcal{T}$. Therefore, even if Σ' is decidable, $\mathcal{T}$ is not decidable. We need more from the axioms to make $\mathcal{T}$ decidable.

Enumerable to decidable

A theory may have axioms that can be organized some way

and we can exploit the organization

to avoid the last mindless search of validity proofs and search them efficiently.

The dedicated search procedures are called decision procedures.

We often show decidability of a theory by providing a decision procedure.

Decidable theories

Definition 20.9

Let $\mathcal{T} = Th(Mod(\Sigma))$. $\mathcal{T}$ is decidable if there is an algorithm that, for each sentence F, can decide whether $F \in \mathcal{T}$ or not.

Definition 20.10 (Equivalent to 20.9)

There is an algorithm that, for each sentence F, can decide whether $\Sigma \Rightarrow F$ or not.

Topic 20.2

Theory Examples

Example decidable and undecidable theories

Example 20.6

Two arithmetics over natural numbers.

Decidable — Presburger [3EXPTIME]:

$$\forall x \neg(x+1=0)$$

$$\forall x \forall y (x+1 = y+1 \Rightarrow x = y)$$

$$F(0) \wedge (\forall x(F(x) \Rightarrow F(x+1)) \Rightarrow \forall x F(x))$$

$$\forall x (x+0 = x)$$

$$\forall x \forall y (x + (y+1) = (x+y)+1)$$

Undecidable — Peano (all of the above, plus):

$$\forall x, y (x \cdot 0 = 0)$$

$$\forall (x \cdot (y+1) = x \cdot y + x)$$

The third axiom is a schema (a pattern of axioms).

Exercise 20.4

Give an algorithm to check if a sentence belongs to the schema or not.

Commentary: A schema is unacceptable if we can not have an algorithm to match the pattern of schema. The story goes that Presburger's PhD advisor was not too pleased by the above decidability result in 1929. The advisor did not know that there will be an area of science, where decidability will be a cornerstone of thinking.

Defining theory

A theory may be expressed in two ways.

1. By giving a set Σ of axioms
2. By giving a set $\mathcal{M}$ of acceptable models

There are theories that can not be expressed by one of the above two ways.

For example,

- Number theory can only be defined using the model. There is no axiomatization. (Due to Gödel's incompleteness theorem)
- Set theory has no "natural" model. We understand set theory via its axioms.

Example: theory of equality $\mathcal{T}_E$

We have treated equality as part of FOL syntax and added special proof rules for it.

We can also treat equality as yet another predicate.

We can encode the behavior of equality as the set of following axioms.

1. $\forall x.\ x = x$
2. $\forall x, y.\ x = y \Rightarrow y = x$
3. $\forall x, y, z.\ x = y \wedge y = z \Rightarrow x = z$
4. for each $f/n \in \mathbf{F}$, $\forall x_1, .., x_n, y_1, .., y_n.\ x_1 = y_1 \wedge .. \wedge x_n = y_n \Rightarrow f(x_1, .., x_n) = f(y_1, .., y_n)$
5. for each $P/n \in \mathbf{R}$, $\forall x_1, .., x_n, y_1, .., y_n.\ x_1 = y_1 \wedge .. \wedge x_n = y_n \wedge P(x_1, .., x_n) \Rightarrow P(y_1, .., y_n)$

The last two axioms are also schema, because they define a set of axioms using a pattern.

Topic 20.3

Fragments/Logics

Decidablity and Complexity

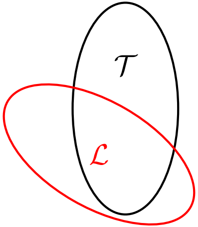
- FOL validity is undecidable
- We restrict the problem in two ways
  1. Theories : limits on the space of models
     - We may get decidable theories
  2. Logics/Fragments : syntactic restrictions on formulas
     - Further reduction in complexity of the decision problem

Fragments

We may restrict $\mathcal{T}$ syntactically to achieve decidablity or low complexity.

Definition 20.11

Let $\mathcal{T}$ be a theory and $\mathcal{L}$ be a set of **S***-sentences. $\mathcal{L}$ wrt $\mathcal{T}$ is decidable if there is an algorithm that takes $F \in \mathcal{L}$ as input and returns if $F \in \mathcal{T}$ or not.*



Sentences

Example 20.7 (Horn clauses)

$\mathcal{L} = \{\forall x.\ A_1(x) \wedge \cdots \wedge A_n(x) \Rightarrow B(x) | A_i \text{ and } B \text{ are atomic}\}$

Quantifier-free fragments

Quantifier-free(QF) fragment has free variables that are assumed to be existentially quantified. (unlike FOL clauses!!)

Often, the quantifier-free fragments of theories have efficient decision procedures.

Example 20.8

The following is a QF formula in the theory of equality

$$f(x) = y \wedge (x = g(a, z) \vee h(x) = g(b))$$

QF of $\mathcal{T}$ of equality has an efficient decision procedure.
Otherwise, the theory is undecidable.

Examples of quantifier-free fragments

Some times the fragments are also referred as logics.

The following important fragments are usually referred using abbreviated names

- quantifier-free theory of equality and uninterpreted function symbols (QF_EUF)
- quantifier-free theory of linear rational arithmetic (QF_LRA)
- quantifier-free theory of uninterpreted function and linear integer arithmetic (QF_UFLIA)

In the following website, find the benchmarks in the above fragments/logics

Visit SMTLIB

`http://smtlib.cs.uiowa.edu/`

Topic 20.4

Problems

Exercise

Exercise 20.5

Prove commutativity of $+$ *in Presburger arithmetic.*

Topic 20.5

Extra slides : complete theories

Complete theory

Definition 20.12

A theory $\mathcal{T}$ is complete if for every sentence F, either $F \in \mathcal{T}$ or $\neg F \in \mathcal{T}$.

Exercise 20.6

When a theory is not complete?

Exercise 20.7

Can a theory have both F and $\neg F$ for some sentence F?

Exercise 20.8

Prove: if $Mod(\mathcal{T})$ is singleton then $\mathcal{T}$ is complete.

A condition for complete theory

Theorem 20.3

If for each $m_1, m_2 \in Mod(\mathcal{T})$ and sentence F,

$$m_1 \models F \text{ iff } m_2 \models F$$

then $\mathcal{T}$ is complete.

Proof.

If F is true in one model in $Mod(\mathcal{T})$ then F is true in all. Therefore, F is complete. □

No sentence can distinguish models of $\mathcal{T}$.

Complete axiomatizable == Decidable

Theorem 20.4

A complete axiomatizable theory is decidable.

Proof.

Since for each **S**-formula F, either F or $\neg F$ is in Σ.

The enumeration in theorem 20.2 will eventually generate proof for F or $\neg F$.

Therefore, complete axiomatizable theory is decidable. □

Decidability via completeness

We can show decidability of a theory via completeness.

We may show completeness as follows.

- there are no finite models
- all countable models are isomorphic (No sentence can distinguish them)

In the previous proof, we enumerate all proofs to look for the members of $\mathcal{T}$.

The method does not tell us about the hardness of the decision problem.

End of Lecture 20