CS228 Logic for Computer Science 2022

Lecture 17: Terms and unification

Instructor: Ashutosh Gupta

IITB, India

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Topic 17.1

Game of terms

CNF formulas and proofs

Example 17.1

Recall we had a proof for $\emptyset \vdash (\forall x.\ (P(x) \lor Q(x)) \Rightarrow \exists x.P(x) \lor \forall x.Q(x))$.

Let us try to prove it via FOL CNF.

We first take negation of the formula and transform it into FOL CNF. We obtain

$$\Sigma \triangleq \{\forall x.\ (P(x) \lor Q(x)), \forall x.\neg P(x), \neg Q(c)\}$$

We have written each clause as a separate formula without dropping quantifiers.

We show that we can derive contradiction from Σ.

CNF formulas and proofs

Recall

$$\Sigma \triangleq \{\forall x.\ (P(x) \lor Q(x)), \forall x. \neg P(x), \neg Q(c)\}$$

Here is a proof that derives contradiction from Σ.

1. $\Sigma \vdash \neg Q(c)$ — Assumption
2. $\Sigma \vdash \forall x.\ (P(x) \lor Q(x))$ — Assumption
3. $\Sigma \vdash P(x) \lor Q(x)$ — $\forall$-Elim applied to 2
4. $\Sigma \vdash \forall x.\ \neg P(x)$ — Assumption
5. $\Sigma \vdash \neg P(x)$ — $\forall$-Elim applied to 4
6. $\Sigma \vdash Q(x)$ — Resolution applied to 3 and 5
7. $\Sigma \vdash \forall x.\ Q(x)$ — $\forall$-Intro applied to 6
8. $\Sigma \vdash Q(c)$ — $\forall$-Elim applied to 7
9. $\Sigma \vdash Q(c) \land \neg Q(c)$ — $\land$-Intro applied to 1 and 8

Step 8 introduced c, which is a non-mechanical step, i.e., we need to plan to choose the term.

Example : an extreme example for finding a magic term.

Example 17.2

Let us derive contradiction from the following.
Let $\Sigma = \{\forall x_4, x_3, x_2, x_1.\ f(x_1, x_3, x_2) \neq f(g(x_2), j(x_4), h(x_3, a))\}$

Let us construct a proof for the above.

1. $\Sigma \vdash \forall x_4, x_3, x_2, x_1.\ f(x_1, x_3, x_2) \neq f(g(x_2), j(x_4), h(x_3, a))$
2. $\Sigma \vdash \forall x_3, x_2, x_1.\ f(x_1, x_3, x_2) \neq f(g(x_2), j(x_4), h(x_3, a))$ *∀-Elim applied to 1*
3. $\Sigma \vdash \forall x_2, x_1.\ f(x_1, j(x_4), x_2) \neq f(g(x_2), j(x_4), h(j(x_4), a))$ *∀-Elim applied to 2*
4. $\Sigma \vdash \forall x_1.\ f(x_1, j(x_4), h(j(x_4), a)) \neq f(g(h(j(x_4), a)), j(x_4), h(j(x_4), a))$ *∀-Elim applied to 3*
5. $\Sigma \vdash f(g(h(j(x_4), a)), j(x_4), h(j(x_4), a)) \neq f(g(h(j(x_4), a)), j(x_4), h(j(x_4), a))$ *∀-Elim applied to 4*

We need a mechanism to auto detect substitutions such that terms with variables become equal

Exercise 17.1

Finish the proof using Reflex and derive contradiction.

How to find the magic terms?

In the previous, example we were asked to equate terms

$$f(x_1, x_3, x_2) \text{ and } f(g(x_2), j(x_4), h(x_3, a))$$

by mapping variables x_1, x_2, x_3, and x_4 to terms.

The process of equating terms is called **unification**.

Sometimes, the unification may not even be possible.

Topic 17.2

Unification

Making terms equal by substitution

Unifier

Definition 17.1

For terms t and u, a substitution σ is a unifier of t and u if $t\sigma = u\sigma$.
We say t and u are unifiable if there is a unifier σ of t and u.

Example 17.3

Find a unifier σ of the following terms

- $x_4\sigma = f(x_1)\sigma$ — $\sigma = \{x_1 \mapsto c, x_4 \mapsto f(c)\}$
- $x_4\sigma = f(x_1)\sigma$ — $\sigma = \{x_1 \mapsto x_2, x_4 \mapsto f(x_2)\}$
- $g(x_1)\sigma = f(x_1)\sigma$ — *not unifiable*
- $x_1\sigma = f(x_1)\sigma$ — *not unifiable*

More general substitution

Commentary: The following definition depends on composition of substitutions, which was discussed in earlier lectures. If not clear please look it up.

Definition 17.2

Let σ_1 and σ_2 be substitutions. σ_1 is more general than σ_2 if there is a substitution τ such that $\sigma_2 = \sigma_1\tau$. We write $\sigma_1 \geq \sigma_2$.

Example 17.4

- $\sigma_1 = \{x \mapsto f(y, z)\} \geq \sigma_2 = \{x \mapsto f(c, g(z)), y \mapsto c, z \mapsto g(z)\}$ *because* $\sigma_2 = \sigma_1\{y \mapsto c, z \mapsto g(z)\}$.
- $\sigma_1 = \{x \mapsto f(y, z)\} \geq \sigma_2 = \{x \mapsto f(z, z), y \mapsto z\}$ *because* $\sigma_2 = \sigma_1\{y \mapsto z\}$.

Exercise 17.2

If $\sigma_1 \geq \sigma_2$ and $\sigma_2 \geq \sigma_3$. Then, $\sigma_1 \geq \sigma_3$.

Most general unifier (mgu)

Is mgu unique? Does mgu always exist?

Definition 17.3

Let t and u be terms with variables, and σ be a unifier of t and u.
σ is most general unifier(mgu) of u and t if it is more general than any other unifier.

Example 17.5

Consider terms $f(x, g(y))$ and $f(g(z), u)$. The following are unifiers of the terms.

1. $\sigma_1 = \{x \mapsto g(z), u \mapsto g(y), z \mapsto z, y \mapsto y\}$
2. $\sigma_2 = \{x \mapsto g(c), u \mapsto g(d), z \mapsto c, y \mapsto d\}$
3. $\sigma_3 = \{x \mapsto g(z), u \mapsto g(z), z \mapsto z, y \mapsto z\}$

where c and d are constants.

Please note $\sigma_1 \geq \sigma_2$ and $\sigma_1 \geq \sigma_3$. $\sigma_2 \not\geq \sigma_3$ and $\sigma_3 \not\geq \sigma_2$.(why?)

Uniqueness of mgu

Definition 17.4

A substitution σ is a renaming if $\sigma : \text{Vars} \rightarrow \text{Vars}$ and σ is one-to-one

Theorem 17.1

If σ_1 and σ_2 are mgus of u and t. Then there is a renaming τ such that $\sigma_1\tau = \sigma_2$.

Proof.

Since σ_1 is mgu, therefore there is a substitution $\hat{\sigma_1}$ such that $\sigma_2 = \sigma_1\hat{\sigma_1}$.
Since σ_2 is mgu, therefore there is a substitution $\hat{\sigma_2}$ such that $\sigma_1 = \sigma_2\hat{\sigma_2}$.
Therefore $\sigma_1 = \sigma_1\hat{\sigma_1}\hat{\sigma_2}$. (And also, $\sigma_2 = \sigma_2\hat{\sigma_2}\hat{\sigma_1}$.)

Without loss of generality, for each $y \in \text{Vars}$, if $y \notin FV(x\sigma_1)$ for each $x \in \text{Vars}$, then we assume $y\hat{\sigma_1} = y$.

Uniqueness of mgu (contd.)

Proof(contd.)

claim: for each $y \in \text{Vars}$, $y\hat{\sigma_1} \in \text{Vars}$

Consider a variable x such that $y \in FV(x\sigma_1)$. Three possibilities for $y\hat{\sigma_1}$.

1. If $y\hat{\sigma_1} = f(..)$, $x\sigma_1\hat{\sigma_1}$ is longer than $x\sigma_1$. Therefore, $x\sigma_1\hat{\sigma_1}\hat{\sigma_2}$ is longer than $x\sigma_1$. Contradiction.
2. If $y\hat{\sigma_1} = c$, $\hat{\sigma_2}$ will not be able to rename c back to y in $x\sigma_1$.
3. Therefore, we must have the third possibility, i.e., $y\hat{\sigma_1} \in \text{Vars}$ is a variable.

claim: for each $y_1 \neq y_2 \in \text{Vars}$, $y_1\hat{\sigma_1} \neq y_2\hat{\sigma_1}$

Assume $y_1\hat{\sigma_1} = y_2\hat{\sigma_1}$. $\hat{\sigma_2}$ will not be able to rename the variables back to distinct variables.(why?)
Contradiction.

$\hat{\sigma_1}$ is a renaming. □

Topic 17.3

Unification algorithm

How to find unifiers?

We need to identify where terms are not in agreement.

Apply substitutions to fix the disagreement.

Disagreement pairs

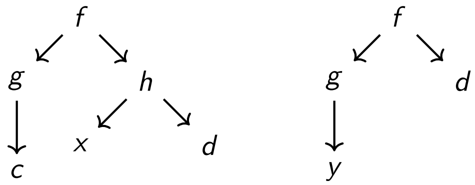
Definition 17.5

For terms t and u, d_1 and d_2 are disagreement pair if

1. *d_1 and d_2 are subterms of t and u respectively,*
2. *the path to d_1 in t is same as and the path to d_2 in u, and*
3. *roots of d_1 and d_2 are different.*

Example 17.6

Consider terms $t = f(g(c), h(x, d))$ and $u = f(g(y), d)$



Disagreement pairs: $h(x, d)$ and d

Disagreement pairs: c and y

Robinson algorithm for computing mgu

Algorithm 17.1: MGU($t, u \in T_S$)

$\sigma := \{\}$;
while $t\sigma \neq u\sigma$ **do**
  choose disagreement pair d_1, d_2 in $t\sigma$ and $u\sigma$;
  if *both* d_1 *and* d_2 *are non-variables* **then** **return** *FAIL* ;
  if $d_1 \in$ Vars **then**
    $x := d_1$; $s := d_2$;
  else
    $x := d_2$; $s := d_1$;
  if $x \in FV(s)$ **then** **return** *FAIL* ;
  $\sigma := \sigma\{x \mapsto s\}$ // update the substitution
return σ

If MGU is sound and always terminates then mgus for unifiable terms always exist.

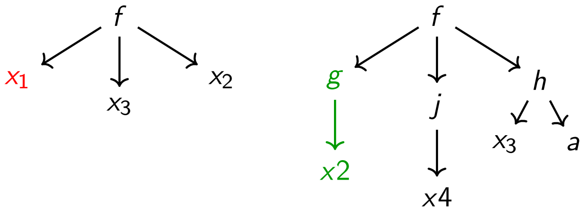
Exercise 17.3

Let $\sigma_0, \sigma_1, \ldots$ be the sequence of observed substitutions during the run of MGU*. Show $\sigma_i \geq \sigma_{i+1}$.*

Example: run of Robinson's algorithm

Example 17.7

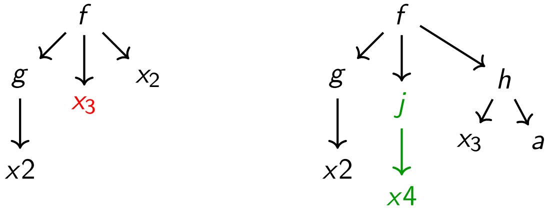
Consider call $\mathrm{MGU}(f(x_1, x_3, x_2), f(g(x_2), j(x_4), h(x_3, a)))$
Initial $\sigma = \{\}$



Disagreement pairs $:= \{\ (x_1, g(x_2)),\ (x_3, j(x_4)),\ (x_2, h(x_3, a))\ \}$
Choose a disagreement pair: $(x_1, g(x_2))$
After update $\sigma = \{x_1 \mapsto g(x_2)\}$
Input terms after applying σ*:* $f(g(x_2), x_3, x_2)$ *and* $f(g(x_2), j(x_4), h(x_3, a))$

Example: run of Robinson's algorithm II (contd.)

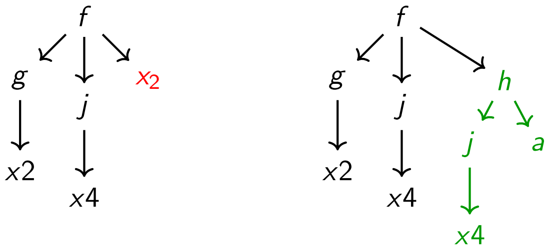
Input terms now:



Disagreement pairs in the new terms:= $\{\ (x_3, j(x_4)),\ (x_2, h(x_3, a))\ \}$
Choose a disagreement pair: $(x_3, j(x_4))$
After update $\sigma = \{x_1 \mapsto g(x_2), x_3 \mapsto j(x_4)\}$
Input terms after applying σ: $f(g(x_2), j(x_4), x_2)$ and $f(g(x_2), j(x_4), h(j(x_4), a))$

Example: run of Robinson's algorithm III(contd.)

Input terms now:



Choose the last disagreement pair: $(x_2, h(j(x_4), a))$.

Since the mapping of x_1 refers to x_2 in old σ, it is also updated.

After applying new mapping $\sigma := \sigma\{x_2 \mapsto h(j(x_4), a)\}$

$$= \{x_1 \mapsto g(x_2), x_3 \mapsto j(x_4)\}\{x_2 \mapsto h(j(x_4), a)\}$$
$$= \{x_1 \mapsto g(h(j(x_4), a)), x_3 \mapsto j(x_4), x_2 \mapsto h(j(x_4), a)\}$$

Terms after applying σ: $f(g(h(j(x_4), a)), j(x_4), h(j(x_4), a))$ and $f(g(h(j(x_4), a)), j(x_4), h(j(x_4), a))$

Since no disagreement pairs, we are done.

Unification in proving

Example 17.8

Consider again $\forall x_1, x_2, x_3, x_4.\ f(x_1, x_3, x_2) \neq f(g(x_2), j(x_4), h(x_3, a))$

Given the above, one may ask

Are $f(x_1, x_3, x_2)$ *and* $f(g(x_2), j(x_4), h(x_3, a))$ *unifiable?*

If we run the unification algorithm on the above terms, we obtain

- $x_1 \mapsto g(h(j(x_4), a))$
- $x_2 \mapsto h(j(x_4), a)$
- $x_3 \mapsto j(x_4)$

We will integrate unification with a simpler resolution proof system.

The above instantiations are not magic anymore!

Topic 17.4

Correctness of Robinson algorithm

Termination of MGU

Theorem 17.2

MGU *always terminates.*

Proof.

Total number of variables in $t\sigma$ and $u\sigma$ decreases in every iteration.(why?)

Since initially there were finite variables in t and u, MGU terminates. □

Soundness of MGU

Theorem 17.3

MGU(t, u) *returns unifier* σ *iff* t *and* u *are unifiable. Furthermore,* σ *is a mgu.*

Proof.

Since MGU must terminate, if t and u are not unifiable then MGU must return FAIL.

Let us suppose t and u are unifiable and τ is a unifier of t and u.
claim: $\tau = \sigma\tau$ is the loop invariant of MGU.

base case:
Initially, σ is identity. Therefore, the invariant holds initially.

induction step:
Induction hypothesis: $\tau = \sigma\tau$ holds at the loop head. ...

Soundness of MGU(contd.)

Proof(contd.)

claim: $t\sigma$ and $u\sigma$ are unifiable.

$$\underbrace{t\sigma\tau = t\tau}_{\text{Ind. Hyp.}} \underbrace{= u\tau}_{\text{Assumption}} \underbrace{= u\sigma\tau}_{\text{Ind. Hyp.}}.$$

claim: $x\tau = s\tau$.

Since $t\sigma\tau = u\sigma\tau$, and x and s are disagreement pairs in $t\sigma$ and $u\sigma$, $x\tau = s\tau$.

claim: $\{x \mapsto s\}\tau = \tau$.

Choose $y \in Vars$.

- If $y = x$, $y\{x \mapsto s\}\tau = s\tau = x\tau = y\tau$.
- If $y \neq x$, $y\{x \mapsto s\}\tau = y\tau$.

Therefore, $\{x \mapsto s\}\tau = \tau$.

Soundness of MGU(contd.)

Proof(contd.)

We now show that if we assume the invariant at the loop head, then FAIL is not returned.

claim: no FAIL at the first if condition
One of d_1 and d_2 is a variable. Otherwise $t\sigma$ and $u\sigma$ are not unifiable.

claim: no FAIL at the last if condition
Since $x\tau = s\tau$, x cannot occurs in s. Otherwise, no unifier can make them equal(why?).

...

Soundness of MGU(contd.)

Proof(contd.)

Since there is no fail, we show that invariant will continue to hold after the iteration.

claim: $\sigma\{x \mapsto s\}\tau = \tau$
Since $\{x \mapsto s\}\tau = \tau$, $\sigma\{x \mapsto s\}\tau = \sigma\tau$. By induction hypothesis, $\sigma\{x \mapsto s\}\tau = \tau$.

Due to the invariant $\tau = \sigma\tau$, σ is mgu at the termination. $\square$

Topic 17.5

Problems

MGU

Exercise 17.4

Find mgu of the following terms

1. $f(g(x_1), h(x_2), x_4)$ *and* $f(g(k(x_2, x_3)), x_3, h(x_1))$
2. $f(x, y, z)$ *and* $f(y, z, x)$
3. $\textsc{mgu}(f(g(x), x), f(y, g(y)))$

Exercise 17.5

Let σ_1 *and* σ_2 *be the MGUs in the above exercise. Give unifiers* σ_1' *and* σ_2' *for the problems respectively such that they are not MGUs. Also give* τ_1 *and* τ_2 *such that*

1. $\sigma_1' = \sigma_1 \tau_1$
2. $\sigma_2' = \sigma_2 \tau_2$

Maximum and minimal substitutions

Exercise 17.6

a. Give two maximum general substitutions *and two minimal general* substitutions*.*

b. Show that maximum general substitutions *are equivalent under renaming.*

Multiple unification

Definition 17.6

Let $t_1, .., t_n$ be terms. A substitution σ is a unifier of $t_1, .., t_n$ if $t_1\sigma = .. = t_n\sigma$.
We say $t_1, .., t_n$ are unifiable if there is a unifier σ of them.

Exercise 17.7

Write an algorithm for computing multiple unifiers using the binary MGU*.*

Concurrent unification

Definition 17.7

Let $t_1, .., t_n$ and $u_1, .., u_n$ be terms. A substitution σ is a concurrent unifier of $t_1, .., t_n$ and $u_1, .., u_n$ if

$$t_1\sigma = u_1\sigma, \quad .., \quad t_n\sigma = u_n\sigma.$$

We say $t_1, .., t_n$ and $u_1, .., u_n$ are concurrently unifiable if there is a unifier σ for them.

Exercise 17.8

Write an algorithm for concurrent unifiers using the binary MGU.

Saturating substitutions

Exercise 17.9

Consider a substitution σ*. Let* $\sigma^1 = \sigma$ *and* $\sigma^{i+1} = \sigma^i\sigma$*. Prove/disprove: for each* σ *there is a number n such that for each number* $k > n$*,* $\sigma^k = \sigma^i$ *for some number* $i \leq n$*.*

Commentary: **Solution:** The property does not hold. Counter example $\sigma = \{x \mapsto f(x)\}$

Topic 17.6

Extra slides: algorithms for unification

Robinson is exponential

Robinson algorithm has worst case exponential run time.

Example 17.9

Consider unification of the following terms
$f(x_1, g(x_1, x_1), x_2, ...)$
$f(g(y_1, y_1), y_2, g(y_2, y_2), ...)$

The mgu:

- $x_1 \mapsto g(y_1, y_1)$
- $y_2 \mapsto g(g(y_1, y_1), g(y_1, y_1))$
- .... *(size of term keeps doubling)*

After discovery of a substitution $x \mapsto s$, Robinson checks if $x \in FV(s)$.
Therefore, Robinson has worst case exponential time.

Martelli-Montanari algorithm

This algorithm is lazy in terms of applying occurs check

Algorithm 17.2: $\text{MM-MGU}(t, u \in T_{\mathbf{S}})$

$\sigma := \lambda x.x;\ M = \{t = u\};$

while *change in M or* σ **do**

  if $f(t_1, ...t_n) = f(u_1, ...u_n) \in M$ **then**

    $M := M \cup \{t_1 = u_1, ...t_n = u_n\} - \{f(t_1, ...t_n) = f(u_1, ...u_n)\};$

  if $f(t_1, ...t_n) = g(u_1, ...u_n) \in M$ **then** **return** *FAIL* ;

  if $x = x \in M$ **then** $M := M - \{x = x\}$;

  if $x = t' \in M$ *or* $t' = x \in M$ **then**

    if $x \in FV(t')$ **then** **return** *FAIL* ;

    $\sigma := \sigma[x \mapsto t'];\ M := M\sigma$

return σ

Commentary: Please find more details on `https://pdfs.semanticscholar.org/3cc3/338b59855659ca77fb5392e2864239c0aa75.pdf`

Escalada-Ghallab Algorithm

There is also Escalada-Ghallab Algorithm for unification.

End of Lecture 17