CS766: Analysis of concurrent programs 2023

Lecture 10: Proof systems for concurrent programs

Instructor: Ashutosh Gupta

IITB, India

Compile date: 2023-02-07

Explicit events analysis is limited

- We have seen analysis of concurrent programs with a bounded set of events
- How do we analyze when we do not have such limits?

We need a proof system.

Commentary: Example and presentation ideas are borrowed from https://fzn.fr/teaching/mpri/2010/fzn-mpri-2010-3.pdf

Topic 10.1

Proof systems for programs

Hoare logic for sequential programs

- Hoare logic is one of the frameworks for the reasoning over programs
- Other logics reason over sets of traces and transitions instead of states
- Can we develop something for concurrent programs?

Proof systems for concurrent programs

- Näive extension of Hoare logic by treating the vector of program counters as a variable
  Not a practical solution(why?)

- Two proof systems that extend Hoare logic for concurrency
  1. Owicki-Gries
  2. Rely-Guarantee (not covered in this lecture)

Topic 10.2

Owicki-Gries proof system

How can we reason over parallel composition?

Commentary: A state formula only refers to variables of a program and does not relate values the variables at different time points.

- Consider all possible interleavings
- Reasoning needs ability to summarize effect of all of them in state formulas

Example 10.1

Assume assignments are atomic

Consider

$$\mathtt{x} := \mathtt{x} + 1 \quad || \quad \mathtt{x} := \mathtt{x} + 2$$

We my conclude : if initially $\mathtt{x} = 0$*, the program finishes with* $\mathtt{x} = 3$*.*

We may write Hoare triple

$$\{\mathtt{x} = 0\} \quad \mathtt{x} := \mathtt{x} + 1 \quad || \quad \mathtt{x} := \mathtt{x} + 2 \quad \{\mathtt{x} = 3\}$$

How can we derive the Hoare triple from the behavior of parts?

Soundness vs completeness

We will design the proof rule for parallel composition.

As we go along, we may be unsound or incomplete, or both.

We will fix those issues in small steps.

Attempt 1: let us model it like nondeterminism (Incomplete and unsound)

$$[\textsc{ParLikeNondet}]\frac{\{P\}c_1\{Q\} \quad \{P\}c_2\{Q\}}{\{P\}c_1||c_2\{Q\}}$$

Example 10.2

$$\frac{\{x = 0\}x := x + 1\{x = 1\} \qquad \{y = 0\}y := y + 1\{y = 1\}}{\{x = y = 0\}x := x + 1||y := y + 1\{x = y = 1\}}\textit{rejected}$$

The rule rejects the above good derivation. It is incomplete.

Example 10.3

$$\frac{\{x = 0\}x := x + 1\{x = 1\} \qquad \{x = 0\}x := x + 1\{x = 1\}}{\{x = 0\}x := x + 1||x := x + 1\{x = 1\}}\ \text{✗}$$

The rule is unsound.

We need to combine the effect of both the programs.

Attempt 2: conjunction of precondition and postcondition (Unsound)

$$[\textsc{ParConjunctive}]\frac{\{P_1\}c_1\{Q_1\} \quad \{P_2\}c_2\{Q_2\}}{\{P_1 \wedge P_2\}c_1||c_2\{Q_1 \wedge Q_2\}}$$

Example 10.4

$$\frac{\{x = 0\}x := x + 1\{x = 1\} \qquad \{y = 0\}y := y + 1\{y = 1\}}{\{x = y = 0\}x := x + 1||y := y + 1\{x = y = 1\}}$$ ✓

Example 10.5

$$\frac{\{y = 1\}x := 1\{y = 1\} \qquad \{\top\}y := 0\{\top\}}{\{y = 1\}x := 1||y := 0\{y = 1\}}$$ ✗

What went wrong? Thread two interfered with truth value of (pre)postcondition of thread one. We need to detect interference.

Attempt 3: monitor interference (Still unsound)

The following condition says that program c does not modify any variable in set of formulas Σ.

$$NoMod(c, \Sigma) \triangleq modifyVars(c) \cap FreeVars(\Sigma) = \emptyset$$

Commentary: We choose FreeVars because we may have quantified formulas in our pre/postcondition

$$[\textsc{ParNoMod}] \frac{\{P_1\}c_1\{Q_1\} \quad \{P_2\}c_2\{Q_2\}}{\{P_1 \wedge P_2\}c_1||c_2\{Q_1 \wedge Q_2\}} NoMod(c_2, \{P_1, Q_1\}) \text{ and } NoMod(c_1, \{P_2, Q_2\})$$

Example 10.6

Is the above rule applicable?

$$\frac{\{z = 0\}x := z; y := x\{y = 0\} \qquad \{\top\}x := 2\{\top\}}{\{z = 0\}x := z; y := x||x := 2\{y = 0\}} \times$$

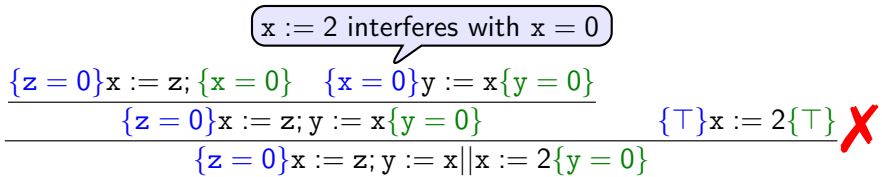
What went wrong? We did not check for interference on intermediate formulas.

We need to detect interference at all intermediate steps.

Example : interference explicated

Example 10.7

Let us look at our example again and write the expanded proof.



Idea: collect intermediate formulas

We modify all proof rules to collect intermediate formulas. For example,

$$[\textsc{Assign}]\frac{}{\{P[exp/x]\}\mathtt{x} := \mathtt{exp}\{P, \{P, P[exp/x]\}\}} \qquad [\textsc{Seq}]\frac{\{P\}\mathtt{c}_1\{Q, \Sigma_1\} \quad \{Q\}\mathtt{c}_2\{R, \Sigma_2\}}{\{P\}\mathtt{c}_1;\mathtt{c}_2\{R, \Sigma_1 \cup \Sigma_2\}}$$

Example 10.8

$$\frac{}{\{\mathtt{x} > 1\}\mathtt{x} := \mathtt{x} - 1\{\mathtt{x} > 0, \{\mathtt{x} > 1, \mathtt{x} > 0\}\}}$$

Exercise 10.1

Write collecting version of all the rules of Hoare logic.

Attempt 4: no interference on collected formulas (Sound, but incomplete)

$$[\textsc{ParNoModCollect}]\ \frac{\{P_1\}c_1\{Q_1,\Sigma_1\} \quad \{P_2\}c_2\{Q_2,\Sigma_2\}}{\{P_1 \wedge P_2\}c_1||c_2\{Q_1 \wedge Q_2, \Sigma_1 \cup \Sigma_2\}}\ \textit{NoMod}(c_2,\Sigma_1) \text{ and } \textit{NoMod}(c_1,\Sigma_2)$$

This proof rule is correct.
But too restrictive

Example 10.9

A good derivation:

$$\frac{\{x>0\}y:=x;\{y>0,\{x>0,y>0\}\} \qquad \{\top\}x:=x+1\{\top,\{\top\}\}}{\{x>0\}y:=x||x:=x+1\{y>0,\{x>0,y>0,\top\}\}}$$

Rejected by the rule!

Because $\textit{NoMod}(x:=x+1,\{x>0,y>0\})$ *is false.*

What went wrong? We went overboard. NoMod is a syntactic check.

Let us make *NoMod* false only if modifications really interfere.

Idea: collect writes

Since only writes interfere, let us collect them explicitly.

We modify all proof rules to collect writes along with intermediate formulas. For example,

$$[\textsc{Assign}]\frac{}{\{P[exp/x]\}\mathtt{x} := \mathtt{exp}\{P, \{P, P[exp/x]\}, \{\mathtt{x} := \mathtt{exp}\}\}}$$

$$[\textsc{Seq}]\frac{\{P\}\mathtt{c}_1\{Q, \Sigma_1, Wrs_1\} \quad \{Q\}\mathtt{c}_2\{R, \Sigma_2, Wrs_2\}}{\{P\}\mathtt{c}_1;\mathtt{c}_2\{R, \Sigma_1 \cup \Sigma_2, Wrs_1 \cup Wars_2\}}$$

Example 10.10

$$\frac{}{\{\mathtt{x} > 0\}\mathtt{y} := \mathtt{x};\{\mathtt{y} > 0, \{\mathtt{x} > 0, \mathtt{y} > 0\}, \{\mathtt{y} := \mathtt{x}\}\}}$$

Exercise 10.2

Write collecting version of all the rules of Hoare logic.

Attempt 5: semantic no interference condition (Still incomplete)

The following condition checks writes in Ws do not interfere invariants in Σ.

$$NoI(Ws, \Sigma) \triangleq \bigwedge_{c \in Ws} \bigwedge_{P \in \Sigma} \{P\}c\{P\} holds$$

$$[\textsc{ParNoInter}] \frac{\{P_1\}c_1\{Q_1, \Sigma_1, Ws_1\} \quad \{P_2\}c_2\{Q_2, \Sigma_2, Ws_2\}}{\{P_1 \wedge P_2\}c_1||c_2\{Q_1 \wedge Q_2, \Sigma_1 \cup \Sigma_2, Ws_1 \cup Ws_2\}} NoI(Ws_2, \Sigma_1) \text{ and } NoI(Ws_1, \Sigma_2)$$

Example 10.11

$$\frac{\{x > 0\}y := x; \{y > 0, \{x > 0, y > 0\}, \{y := x\}\} \qquad \{\top\}x := x + 1\{\top, \{\top\}, \{x := x + 1\}\}}{\{x > 0\}y := x||x := x + 1\{y > 0, \{x > 0, y > 0, \top\}, \{y := x, x := x + 1\}\}} \checkmark$$

Are we done?

Not quite.

Example 10.12

Consider the following correct derivation which is disallowed by [ParNoInter].

$$\frac{\{x > 1\}y := x;\{y > 1, \{x > 1, y > 1\}, \{y := x\}\} \qquad \{x > 3\}x := x - 1\{\top, \{x > 3\}, \{x := x - 1\}\}}{\{x > 3\}y := x || x := x - 1\{y > 1, \{...\}, \{...\}\}}$$

The derivation is not possible because

$NoI(\{x := x - 1\}, \{x > 1, y > 1\})$

$= \underbrace{\{x > 1\}x := x - 1\{x > 1\}}_{\text{Does not hold}}$ and $\{y > 1\}x := x - 1\{y > 1\} = False$

We are not complete. We are still rejecting good proofs.

How to find a weaker rule, while preserving soundness?

Idea: collect writes with context

We modify [ASSIGN] rule again to collect writes with their contexts. For example,

$$[\text{ASSIGN}]\frac{}{\{P[exp/x]\}\mathtt{x} := \mathtt{exp}\{P, \{P, P[exp/x]\}, \{\ \{P[exp/x]\}\mathtt{x} := \mathtt{exp}\ \}\}}$$

We also need to modify [HAVOC]. Rest remains the same.

Example 10.13

$$\frac{}{\{\mathtt{x} > 0\}\mathtt{y} := \mathtt{z}; \{\mathtt{x} > 0, \{\mathtt{x} > 0\}, \{\ \{\mathtt{x} > 0\}\mathtt{y} := \mathtt{z}\ \}\}}$$

Write with the condition under which it executes.

Attempt 6: Owicki-Gries proof rule (Sound and complete)

The following condition checks writes in Ws do not interfere invariants in Σ.

$$NoInter(Ws, \Sigma) \triangleq \bigwedge_{\{Q\}c \in Ws} \bigwedge_{P \in \Sigma} \{P \wedge Q\}c\{P\} holds$$

$$[\textsc{Par}] \frac{\{P_1\}c_1\{Q_1, \Sigma_1, Ws_1\} \quad \{P_2\}c_2\{Q_2, \Sigma_2, Ws_2\}}{\{P_1 \wedge P_2\}c_1||c_2\{Q_1 \wedge Q_2, \Sigma_1 \cup \Sigma_2, Ws_1 \cup Ws_2\}} NoInter(Ws_2, \Sigma_1) \text{ and } NoInter(Ws_1, \Sigma_2)$$

Example: interference checking with context

Example 10.14

$$\frac{\{x > 1\}y := x; \{y > 1, \{x > 1, y > 1\}, \{\{x > 1\}y := x\}\} \quad \{x>3\}x := x-1\{\top, \{x > 3\}, \{\{x > 3\}x := x-1\}\}}{\{x > 3\}y := x || x := x - 1\{y > 1, \{...\}, \{...\}\}}$$

The above derivation is acceptable by the PAR *rule because the side conditions are satisfied.*

$$\begin{aligned} \mathit{NoInter}(\mathit{Ws}_2, \Sigma_1) &= \mathit{NoInter}(\{\{x > 3\}x := x - 1\}, \{x > 1, y > 1\}) \\ &= \{x > 1 \wedge x > 3\}x := x - 1\{x > 1\} \text{ and } \{y > 1 \wedge x > 3\}x := x - 1\{y > 1\} = \top \end{aligned}$$

Exercise 10.3

Show $\mathit{NoInter}(\mathit{Ws}_1, \Sigma_2)$ *is true.*

Commentary: Please check if this proof matches with the earlier proof-rule like notation.

Example: let us prove a program

Let us prove.

$$\{x = 0\}x := x + 1||x := x + 2\{x = 3\}$$

Let us display the Owicki-Gries proof in a more convenient notation

$$\{x = 0\}$$

$$\begin{array}{l} \{P_1 : x = 0 \vee x = 2\} \\ x := x + 1; \\ \{Q_1 : x = 1 \vee x = 3\} \end{array} \quad || \quad \begin{array}{l} \{P_2 : x = 0 \vee x = 1\} \\ x := x + 2; \\ \{Q_2 : x = 2 \vee x = 3\} \end{array}$$

$$\{x = 3\}$$

Noninterference checks:

- $\{P_2 \wedge P_1\}x := x + 1\{P_2\}$
- $\{Q_2 \wedge P_1\}x := x + 1\{Q_2\}$
- $\{P_1 \wedge P_2\}x := x + 2\{P_1\}$
- $\{Q_1 \wedge P_2\}x := x + 2\{Q_1\}$

Exercise 10.4

a. Check $x = 0 \Rightarrow P_1 \wedge P_2$ *and* $Q_1 \wedge Q_2 \Rightarrow x = 3$. *b. Check noninterference checks.*

Example: let us prove one more

Let us suppose we need to prove.

$$\{x = 0\}x := x + 1||x := x + 1\{x = 2\}$$

Here is a Owicki-Gries proof.

$$\{x = 0 \wedge pc_1 = 0 \wedge pc_2 = 0\}$$

$$\begin{array}{l|l} \{pc_1 = 0 \wedge (pc_2 = 0 \Rightarrow x = 0) \wedge (pc_2 = 1 \Rightarrow x = 1)\} & \{pc_2 = 0 \wedge (pc_1 = 0 \Rightarrow x = 0) \wedge (pc_1 = 1 \Rightarrow x = 1)\} \\ x := x + 1; pc_1 := 1; & x := x + 1; pc_2 := 1; \\ \{pc_1 = 1 \wedge (pc_2 = 0 \Rightarrow x = 1) \wedge (pc_2 = 1 \Rightarrow x = 2)\} & \{pc_2 = 1 \wedge (pc_1 = 0 \Rightarrow x = 1) \wedge (pc_1 = 1 \Rightarrow x = 2)\} \end{array}$$

$$\{x = 2\}$$

Noninterference check remain the same. Please verify!

Locals may appear in the proof of the other thread.

Thread modular proofs

Definition 10.1

An Owicki-Gries proof is thread modular if the proof of a thread only refer to its locals and the globals.

Proofs are not thread modular, when globals lack information to describe the invariants.

Example 10.15

In a mutual exclusion protocol, if globals do not record who has the lock, then we need to refer to program counters of threads in the proofs.

Non-thread modular proofs tend to be cumbersome. As a principle, it is desirable to minimize reference to the locals of other threads.

Another example: proving victim mutual exclusion

$\{\top\}$

$\{P_1 : \top\}$		$\{P_2 : \top\}$
$0 : victim = 0;$		$0 : victim = 1;$
$\{Q_1 : (\mathrm{pc}_2 \neq 1 \Rightarrow victim = 0)\}$		$\{Q_2 : (\mathrm{pc}_1 \neq 1 \Rightarrow victim = 1)\}$
$1 :$ `while`$(victim == 0);$	$\|\|$	$1 :$ `while`$(victim == 1);$
$\{R_1 : \mathrm{pc}_1 = 2 \wedge \mathrm{pc}_2 = 1 \wedge victim = 1\}$		$\{R_2 : \mathrm{pc}_2 = 2 \wedge \mathrm{pc}_1 = 1 \wedge victim = 0\}$
$2 :$ //critical section		$2 :$ //critical section

$\{\bot\}$

Says both the threads cannot finish

Some noninterference checks for thread 1 invariants against thread 2 writes:

- No write can interfere with P_1, since it is $\top$.
- $\{Q_1 \wedge \top\}\mathrm{pc}_2 = 0 \wedge victim' = 1 \wedge \mathrm{pc}_2' = 1\{Q_1\}$
- $\{Q_1 \wedge (\mathrm{pc}_1 \neq 1 \Rightarrow victim = 1)\}\underbrace{\mathrm{pc}_2 = 1 \wedge victim = 0 \wedge victim = victim' \wedge \mathrm{pc}_2' = 2;}_{\text{Exit branch of the loop in the second thread}}\{Q_1\}$

Exercise 10.5

a. Show R_1, P_2, Q_2, R_2 are free from interference.

b. How many noninterference checks are needed?

Since exit from the loop modifies pc_2, we need to check the formulas that mention it.

End of Lecture 10