

# CS228 Logic for Computer Science 2023

## Lecture 5: Formal proofs - derived rules

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# Derived rules

In logical thinking, we have many deductions that are **not listed** in our rules.

The deductions are consequence of our rules. We call them **derived rules**.

Let us look at a few.

## Topic 5.1

Derived rules: unit resolution, tautology, contradiction, contrapositive

## Derived rules : unit resolution

### Theorem 5.1

If we have  $\Sigma \vdash \neg F \vee G$  and  $\Sigma \vdash F$ , we can derive  $\Sigma \vdash G$ .

Proof.

1.  $\Sigma \vdash \neg F \vee G$  Premise
2.  $\Sigma \vdash F$  Premise
3.  $\Sigma \vdash F \Rightarrow G$   $\Rightarrow$ -Def applied to 1
4.  $\Sigma \vdash G$   $\Rightarrow$ -Elim applied to 2 and 3

□

We can use the above derivation as a **sub-procedure** and introduce the following proof rule.

$$\text{UNITRES} \frac{\Sigma \vdash \neg F \vee G \quad \Sigma \vdash F}{\Sigma \vdash G}$$

## Example: implication

### Example 5.1

Let us prove  $\{(\neg p \vee r), (p \vee \neg q)\} \vdash (q \Rightarrow p \wedge r)$ .

- |   |   |
|---|---|
| 1. $\{(\neg p \vee r), (p \vee \neg q)\} \cup \{q\} \vdash q$               | <i>Assumption</i>                                   |
| 2. $\{(\neg p \vee r), (p \vee \neg q)\} \cup \{q\} \vdash (p \vee \neg q)$ | <i>Assumption</i>                                   |
| 3. $\{(\neg p \vee r), (p \vee \neg q)\} \cup \{q\} \vdash (\neg q \vee p)$ | <i><math>\vee</math>-Symm applied to 2</i>          |
| 4. $\{(\neg p \vee r), (p \vee \neg q)\} \cup \{q\} \vdash p$               | <i>UnitRes applied to 1 and 3</i>                   |
| 5. $\{(\neg p \vee r), (p \vee \neg q)\} \cup \{q\} \vdash (\neg p \vee r)$ | <i>Assumption</i>                                   |
| 6. $\{(\neg p \vee r), (p \vee \neg q)\} \cup \{q\} \vdash r$               | <i>UnitRes applied to 4 and 5</i>                   |
| 7. $\{(\neg p \vee r), (p \vee \neg q)\} \cup \{q\} \vdash p \wedge r$      | <i><math>\wedge</math>-Intro applied to 4 and 6</i> |
| 8. $\{(\neg p \vee r), (p \vee \neg q)\} \vdash (q \Rightarrow p \wedge r)$ | <i><math>\Rightarrow</math>-Intro applied to 7</i>  |

I run when it rains or when it does not.

A convoluted way of saying something is always true.

## Derived rules: tautology rule

### Theorem 5.2

For any  $F$  and a set  $\Sigma$  of formulas, we can always derive  $\Sigma \vdash \neg F \vee F$ .

Proof.

1.  $\Sigma \cup \{F\} \vdash F$
2.  $\Sigma \vdash F \Rightarrow F$
3.  $\Sigma \vdash \neg F \vee F$

Assumption

$\Rightarrow$ -Intro applied to 1

$\Rightarrow$ -Def applied to 2

□

Again, we can introduce the following proof rule.

$$\text{TAUTOLOGY} \frac{}{\Sigma \vdash \neg F \vee F}$$

## Contradiction

If I eat a cake and **not** eat it, then **sun is cold**.

Once we introduce **an absurdity** (formally contradiction), there are **no limits** in absurdity.

**Commentary:** To explain the importance of logic. Once Bertrand Russell made the following argument,  
1.  $2+2 = 5$    2.  $4=5$    3.  $4-3 = 5-3$    4.  $1=2$    5. Pope and I are two.   6. Pope and I are one.   6. I am Pope.



## Derived rules: contradiction rule

### Theorem 5.3

If we have  $\Sigma \vdash F \wedge \neg F$ , we can always derive  $\Sigma \vdash G$ .

Proof.

1.  $\Sigma \vdash F \wedge \neg F$

Premise

2.  $\Sigma \vdash \neg F \wedge F$

$\wedge$ -Symm applied to 1

3.  $\Sigma \vdash \neg F$

$\wedge$ -Elim applied to 2

4.  $\Sigma \vdash \neg F \vee G$

$\vee$ -Intro applied to 3

5.  $\Sigma \vdash F$

$\wedge$ -Elim applied to 1

6.  $\Sigma \vdash G$

UnitRes applied to 4 and 5

□

Therefore, we may declare the following derived proof rule

$$\text{CONTRA} \frac{\Sigma \vdash F \wedge \neg F}{\Sigma \vdash G}$$

I think, therefore I am. -Descartes



I am not, therefore I do not think.

In an argument, negation of the conclusion implies negation of premise.

## Derived rules: contrapositive rule

### Theorem 5.4

If we have  $\Sigma \cup \{F\} \vdash G$ , we can always derive  $\Sigma \cup \{\neg G\} \vdash \neg F$ .

Proof.

- |   |                                   |  |  |
|---|-----------------------------------|--|--|
| 1. $\Sigma \cup \{F\} \vdash G$             | Premise                           | 6. $\Sigma \vdash (\neg G \Rightarrow \neg F)$                 | $\Rightarrow$ -Def applied to 5        |
| 2. $\Sigma \cup \{F\} \vdash \neg\neg G$    | DoubleNeg applied to 1            | 7. $\Sigma \cup \{\neg G\} \vdash (\neg G \Rightarrow \neg F)$ | Monotonic applied to 6                 |
| 3. $\Sigma \vdash F \Rightarrow \neg\neg G$ | $\Rightarrow$ -Intro applied to 2 | 8. $\Sigma \cup \{\neg G\} \vdash \neg F$                      | Assumption                             |
| 4. $\Sigma \vdash \neg F \vee \neg\neg G$   | $\Rightarrow$ -Def applied to 3   | 9. $\Sigma \cup \{\neg G\} \vdash \neg F$                      | $\Rightarrow$ -Elim applied to 7 and 8 |
| 5. $\Sigma \vdash \neg\neg G \vee \neg F$   | $\vee$ -Symm applied to 4         |  |  |

□

Therefore, we may declare the following derived proof rule

$$\text{CONTRAPOSITIVE} \frac{\Sigma \cup \{F\} \vdash G}{\Sigma \cup \{\neg G\} \vdash \neg F}$$

## Topic 5.2

More derived rules: proof by cases and contradiction, reverse double negation, and resolution

# Proof by cases and contradiction

We must have seen the following proof structure

## ► Proof by cases

If I have money, I run.  
If I do not have money, I run.  
Therefore, I run.

## ► Proof by contradiction

Assume, I ate a dinosaur.  
My tummy is far smaller than a dinosaur. **Contradiction.**  
Therefore, I did not eat dinosaur.

## Derived rules: proof by cases

### Theorem 5.5

*If we have  $\Sigma \cup \{F\} \vdash G$  and  $\Sigma \cup \{\neg F\} \vdash G$ , we can always derive  $\Sigma \vdash G$ .*

Proof.

1.  $\Sigma \cup \{F\} \vdash G$  Premise
2.  $\Sigma \cup \{\neg F\} \vdash G$  Premise
3.  $\Sigma \vdash F \vee \neg F$  Tautology
4.  $\Sigma \vdash G$  V-Elim applied to 1,2, and 3

□

Therefore, we may declare the following derived proof rule

$$\text{BYCASES} \frac{\Sigma \cup \{F\} \vdash G \quad \Sigma \cup \{\neg F\} \vdash G}{\Sigma \vdash G}$$

## Derived rules: proof by contradiction

### Theorem 5.6

*If we have  $\Sigma \cup \{F\} \vdash G$  and  $\Sigma \cup \{F\} \vdash \neg G$ , we can always derive  $\Sigma \vdash \neg F$ .*

Proof.

1.  $\Sigma \cup \{F\} \vdash G$  Premise
2.  $\Sigma \cup \{F\} \vdash \neg G$  Premise
3.  $\Sigma \cup \{\neg G\} \vdash \neg F$  Contrapositive applied to 1
4.  $\Sigma \cup \{\neg\neg G\} \vdash \neg F$  Contrapositive applied to 2
5.  $\Sigma \vdash \neg F$  ByCases 3 and 4

□

Therefore, we may declare the following derived proof rule

$$\text{BYCONTRA} \frac{\Sigma \cup \{F\} \vdash G \quad \Sigma \cup \{F\} \vdash \neg G}{\Sigma \vdash \neg F}$$

## Reverse double negation

I do not dislike apples.

Therefore, I like apples.



## Derived rule: reverse double negation

### Theorem 5.7

If we have  $\Sigma \vdash \neg\neg F$ , we can always derive  $\Sigma \vdash F$ .

Proof.

1.  $\Sigma \vdash \neg\neg F$  Premise
2.  $\Sigma \cup \{\neg F\} \vdash \neg\neg F$  Monotonic applied to 1
3.  $\Sigma \cup \{\neg F\} \vdash \neg F$  Assumption
4.  $\Sigma \cup \{\neg F\} \vdash \neg F \wedge \neg\neg F$   $\wedge$ -Intro applied to 2 and 3
5.  $\Sigma \cup \{\neg F\} \vdash F$  Contra applied to 4
6.  $\Sigma \cup \{F\} \vdash F$  Assumption
7.  $\Sigma \vdash F$  Proof by cases applied to 5 and 6  $\square$

Therefore, we may declare the following derived proof rule

$$\text{REVDOUBLENEG} \frac{\Sigma \vdash \neg\neg F}{\Sigma \vdash F}$$

I ate or ran. I did not eat or sleep.

Therefore, I ran or sleep.

## Derived rules : resolution

### Theorem 5.8

*If we have  $\Sigma \vdash \neg F \vee G$  and  $\Sigma \vdash F \vee H$ , we can derive  $\Sigma \vdash G \vee H$ .*

Proof.

1.  $\Sigma \vdash \neg F \vee G$
2.  $\Sigma \cup \{F\} \vdash \neg F \vee G$
3.  $\Sigma \cup \{F\} \vdash F$
4.  $\Sigma \cup \{F\} \vdash G$
5.  $\Sigma \cup \{F\} \vdash G \vee H$

Premise  
Monotonic applied to 1  
Assumption  
UnitRes applied to 2 and 3  
 $\vee$ -Intro applied to 4 } Case 1

...

## Derived rules : resolution (contd.)

### Proof(contd.)

6. $\Sigma \vdash F \vee H$	Premise	} Substitution from $F$ to $\neg\neg F$
7. $\Sigma \cup \{F\} \vdash \neg\neg F$	DoubleNeg applied to 3	
8. $\Sigma \cup \{F\} \vdash \neg\neg F \vee H$	$\vee$ -Intro applied to 7	
9. $\Sigma \cup \{H\} \vdash H$	Assumption	
10. $\Sigma \cup \{H\} \vdash H \vee \neg\neg F$	$\vee$ -Intro applied to 9	
11. $\Sigma \cup \{H\} \vdash \neg\neg F \vee H$	$\vee$ -Symm applied to 10	
12. $\Sigma \vdash \neg\neg F \vee H$	$\vee$ -Elim applied to 6, 8, and 11	

...

## Derived rules : resolution (contd.)

Proof(contd.)

$$13. \Sigma \cup \{\neg F\} \vdash \neg\neg F \vee H$$

$$14. \Sigma \cup \{\neg F\} \vdash \neg F$$

$$15. \Sigma \cup \{\neg F\} \vdash H$$

$$16. \Sigma \cup \{\neg F\} \vdash H \vee G$$

$$17. \Sigma \cup \{\neg F\} \vdash G \vee H$$

Monotonic applied to 12  
Assumption  
UnitRes applied to 13 and 14  
 $\vee$ -Intro applied to 15  
 $\vee$ -Symm applied to 16

} Case 2

$$18. \Sigma \vdash G \vee H$$

Proof by cases applied to 5 and 17



Therefore, we may declare the following derived proof rule

$$\text{RESOLUTION} \frac{\Sigma \vdash F \vee G \quad \Sigma \vdash \neg F \vee H}{\Sigma \vdash G \vee H}$$

## Topic 5.3

### Substitution and formal proofs

# Derivations for substitutions

## Theorem 5.9

Let  $F_1(p)$  and  $F_2(p)$  be formulas. If we have  $\Sigma \vdash F_1(G) \Leftrightarrow F_1(H)$ ,  $\Sigma \vdash F_2(G) \Leftrightarrow F_2(H)$ , and  $\Sigma \vdash F_1(G) \wedge F_2(G)$ , we can derive  $\Sigma \vdash F_1(H) \wedge F_2(H)$ .

Proof.

- |  |  |  |  |
|--|--|--|--|
| 1. $\Sigma \vdash F_1(G) \Leftrightarrow F_1(H)$ | Premise                                | 7. $\Sigma \vdash F_2(G) \wedge F_1(G)$      | $\wedge$ -Symm applied to 3            |
| 2. $\Sigma \vdash F_2(G) \Leftrightarrow F_2(H)$ | Premise                                | 8. $\Sigma \vdash F_2(G)$                    | $\wedge$ -Elim applied to 7            |
| 3. $\Sigma \vdash F_1(G) \wedge F_2(G)$          | Premise                                | 9. $\Sigma \vdash F_2(G) \Rightarrow F_2(H)$ | $\Leftrightarrow$ -Def applied to 2    |
| 4. $\Sigma \vdash F_1(G)$                        | $\wedge$ -Elim applied to 3            | 10. $\Sigma \vdash F_2(H)$                   | $\Rightarrow$ -Elim applied to 8 and 9 |
| 5. $\Sigma \vdash F_1(G) \Rightarrow F_1(H)$     | $\Leftrightarrow$ -Def applied to 1    | 11. $\Sigma \vdash F_1(H) \wedge F_2(H)$     | $\wedge$ -Intro applied to 6 and 10    |
| 6. $\Sigma \vdash F_1(H)$                        | $\Rightarrow$ -Elim applied to 4 and 5 |  |  |

□

## Exercise 5.1

We have proven the above theorem for  $\wedge$ . Write similar proofs for  $\vee$ ,  $\neg$ ,  $\Rightarrow$ ,  $\oplus$ , and  $\Leftrightarrow$ .

## Substitution rule

### Theorem 5.10

*Let  $F(p)$  be a formula. If we have  $\Sigma \vdash G \Leftrightarrow H$  and  $\Sigma \vdash F(G)$ , we can derive  $\Sigma \vdash F(H)$ .*

### Proof.

Using theorems like theorem 5.9 for each connective, we can build an induction argument for the above. □

# We shall not introduce substitution as a rule.

### Exercise 5.2

*Write the inductive proof for the above theorem.*

**Commentary:** The above theorem is not like other theorems in this lecture. Replacing  $F(G)$  by  $F(H)$  causes long range changes in the formula. Considering such transformation as a unit step in a proof is not ideal. Ideally, we should be able to check a proof step in constant time. We need linear time in terms of formula size to check a proof step due to substitution. Some theorem provers allow substitution as a single step. In this course, we will not.



## Example: disallowed substitution operation

### Example 5.2

*The following proof step is not allowed in our proof system.*

1.  $\Sigma \vdash \neg(\neg\neg F \vee G)$

2.  $\Sigma \vdash \neg(F \vee G)$

.....  
*RevDoubleNeg applied to  $\neg\neg F$  in 1*

*We can apply transformations only on the top formulas.*

### Exercise 5.3

*Write an acceptable version of the above derivation.*

**Commentary:** In the proof of resolution rule, we needed a similar shortcut when we needed to derive statement  $\Sigma \vdash \neg\neg F \vee H$  from  $\Sigma \vdash F \vee H$ . We spent 5-6 steps to derive the statement.

## Topic 5.4

Motivate next lecture

# Mathematics vs. computer science

So far we saw rules of reasoning.

We have seen that the rules are correct and will see in a few lectures that they are also **sufficient**, i.e., all true statements are derivable.

Our inner mathematician is happy!!

However, our **inner computer scientist is unhappy**.

- ▶ Too many rules - dozens of rules
- ▶ No instructions (or algorithm) for applying them on a given problem

We will embark upon simplifying and automating the reasoning process.

# Topic 5.5

## Problems

# Formal proofs

## Exercise 5.4

*Derive the following statements*

1.  $\{(p \Rightarrow q), (p \vee q)\} \vdash q$
2.  $\{(p \Rightarrow q), (q \Rightarrow r)\} \vdash \neg(\neg r \wedge p)$
3.  $\{(q \vee (r \wedge s)), (q \Rightarrow t), (t \Rightarrow s)\} \vdash s$
4.  $\{(p \vee q), (r \vee s)\} \vdash ((p \wedge r) \vee q \vee s)$
5.  $\{(((p \Rightarrow q) \Rightarrow q) \Rightarrow q)\} \vdash (p \Rightarrow q)$
6.  $\emptyset \vdash (p \Rightarrow (q \vee r)) \vee (r \Rightarrow \neg p)$
7.  $\{p\} \vdash (q \Rightarrow p)$
8.  $\{(p \Rightarrow (q \Rightarrow r))\} \vdash ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$
9.  $\{(\neg p \Rightarrow \neg q)\} \vdash (q \Rightarrow p)$
10.  $\{r \vee (s \wedge \neg t), (r \vee s) \Rightarrow (u \vee \neg t)\} \vdash t \Rightarrow u$

End of Lecture 5