CS 433 Automated Reasoning 2025

Lecture 6: SAT Solvers

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Propositional satisfiability problem

Consider a propositional logic formula F.

Find a model m such that

$$m \models F.$$

Example 6.1

Give a model of $p_1 \wedge (\neg p_2 \vee p_3)$

Some terminology

- Propositional variables are also referred as atoms
- A literal is either an atom or its negation
- A clause is a disjunction of literals.

Since $\vee$ is associative, commutative, and absorbs multiple occurrences, a clause may be referred as a set of literals

Example 6.2

- *p is an atom but $\neg p$ is not.*
- *$\neg p$ and p both are literals.*
- *$p \vee \neg p \vee p \vee q$ is a clause.*
- *$\{p, \neg p, q\}$ is the same clause.*

Conjunctive normal form(CNF)

Definition 6.1

A formula is in CNF if it is a conjunction of clauses.

Since $\wedge$ is associative, commutative, and absorbs multiple occurrences, a CNF formula may be referred as a set of clauses

Example 6.3

- $\neg p$ *and* p *both are in CNF.*
- $(p \vee \neg q) \wedge (r \vee \neg q) \wedge \neg r$ *in CNF.*
- $\{(p \vee \neg q), (r \vee \neg q), \neg r\}$ *is the same CNF formula.*
- $\{\{p, \neg q\}, \{r, \neg q\}, \{\neg r\}\}$ *is the same CNF formula.*

Exercise 6.1

Write a formal grammar for CNF

CNF input

We assume that the input formula to a SAT solver is always in CNF.

Tseitin encoding can convert each formula into a CNF without any blowup.

- introduces fresh variables

Topic 6.1

DPLL (Davis-Putnam-Loveland-Logemann) method

Notation: partial model

Definition 6.2

We will call elements of $\mathbf{Vars} \hookrightarrow \mathcal{B}$ *as partial models.*

Notation: state of a literal

Under partial model m,

a literal ℓ is true if $m(\ell) = 1$ and
ℓ is false if $m(\ell) = 0$.

Otherwise, ℓ is unassigned.

Exercise 6.2

Consider partial model $m = \{p_1 \mapsto 0, p_2 \mapsto 1\}$

What are the states of the following literals under m?

- p_1
- p_2
- p_3
- $\neg p_1$

Notation: state of a clause

Under partial model m,

a clause C is true if there is $\ell \in C$ such that ℓ is true and
C is false if for each $\ell \in C$, ℓ is false.

Otherwise, C is unassigned.

Exercise 6.3

Consider partial model $m = \{p_1 \mapsto 0, p_2 \mapsto 1\}$

What are the states of the following clauses under m?

- $p_1 \vee p_2 \vee p_3$
- $p_1 \vee \neg p_2$
- $p_1 \vee p_3$
- $\emptyset$ *(empty clause)*

Notation: state of a formula

Under partial model m,

CNF F is true if for each $C \in F$, C is true and
F is false if there is $C \in F$ such that C is false.

Otherwise, F is unassigned.

Exercise 6.4

Consider partial model $m = \{p_1 \mapsto 0, p_2 \mapsto 1\}$

What are the states of the following formulas under m*?*

- $(p_3 \vee \neg p_1) \wedge (p_1 \vee \neg p_2)$
- $(p_1 \vee p_2 \vee p_3) \wedge \neg p_1$
- $p_1 \vee p_3$
- $\emptyset$ *(empty formula)*

Notation: unit clause and unit literal

Definition 6.3

C is a unit clause under m if a literal $\ell \in C$ is unassigned and the rest are false. ℓ is called unit literal.

Exercise 6.5

Consider partial model $m = \{p_1 \mapsto 0, p_2 \mapsto 1\}$

Are the following clauses unit under m? If yes, please identify the unit literals.

- $p_1 \lor \neg p_3 \lor \neg p_2$
- $p_1 \lor \neg p_3 \lor p_2$
- $p_1 \lor \neg p_3 \lor p_4$
- $p_1 \lor \neg p_2$

DPLL (Davis-Putnam-Loveland-Logemann) method

DPLL

- maintains a partial model, initially $\emptyset$
- assigns unassigned variables 0 or 1 randomly one after another
- sometimes forced to choose assignments due to unit literals(Why?)

DPLL

Algorithm 6.1: DPLL(F)

Input: CNF F **Output:** sat/unsat

return $DPLL(F, \emptyset)$

Algorithm 6.2: DPLL(F,m)

Input: CNF F, partial assignment m **Output:** sat/unsat

if *F is true under m* **then** **return** *sat* ;

if *F is false under m* **then** **return** *unsat* ;

Backtracking at conflict

if *$\exists$ unit literal p under m* **then**

  return $DPLL(F, m[p \mapsto 1])$

if *$\exists$ unit literal $\neg p$ under m* **then**

  return $DPLL(F, m[p \mapsto 0])$

Unit propagation

Decision

Choose an unassigned variable p and a random bit $b \in \{0, 1\}$;

if $DPLL(F, m[p \mapsto b]) == sat$ **then**

  return *sat*

else

  return $DPLL(F, m[p \mapsto 1 - b])$

Three actions of DPLL

A DPLL run consists of three types of actions

- Decision
- Unit propagation
- Backtracking

Exercise 6.6

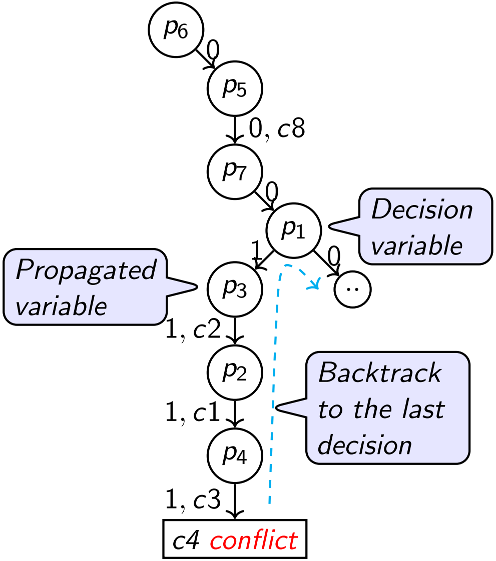
What is the worst case complexity of DPLL?

Example: decide, propagate, and backtrack in DPLL

Example 6.4

$$c_1 = (\neg p_1 \vee p_2)$$
$$c_2 = (\neg p_1 \vee p_3 \vee p_5)$$
$$c_3 = (\neg p_2 \vee p_4)$$
$$c_4 = (\neg p_3 \vee \neg p_4)$$
$$c_5 = (p_1 \vee p_5 \vee \neg p_2)$$
$$c_6 = (p_2 \vee p_3)$$
$$c_7 = (p_2 \vee \neg p_3 \vee p_7)$$
$$c_8 = (p_6 \vee \neg p_5)$$

Blue : causing unit propagation
Green/Blue : true clause



Exercise 6.7 *Complete the DPLL run*

Optimizations

DPLL allows many optimizations.

We will discuss many optimizations.

- **clause learning**
- 2-watched literals
- ...

First, let us look at a revolutionary optimization.

Topic 6.2

Clause learning

Clause learning

As we decide and propagate,

we construct a data structure, called implication graph, to

observe the run and avoid unnecessary backtracking.

Notation: run of DPLL

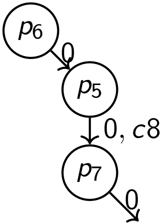
Some notation before we introduce implication graph.

Definition 6.4

We call the current partial model a run of DPLL.

Example 6.5

Borrowing from the earlier example, the following is a run that has not reached to the conflict yet.

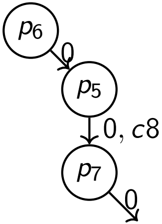


Notation: Decision level

Definition 6.5

During a run, the decision level of a true literal is the number of decisions after which the literal was made true.

Example 6.6



Given the run, we write $\neg p_5 @1$ *to indicate that* $\neg p_5$ *was set to true after one decision.*

Similarly, we write $\neg p_7 @2$ *and* $\neg p_6 @1$.

Implication graph

During the DPLL run, we maintain the following data structure.

Definition 6.6

Under a partial model m, the implication graph is a labeled DAG (N, E), where

- *N is the set of true literals under m and a conflict node*
- $E = \{(\ell_1, \ell_2) | \neg\ell_1 \in causeClause(\ell_2) \text{ and } \ell_2 \neq \neg\ell_1\}$

$$causeClause(\ell) \triangleq \begin{cases} \text{clause due to which unit propagation made } \ell \text{ true} \\ \emptyset \text{ for the literals of the decision variables} \end{cases}$$

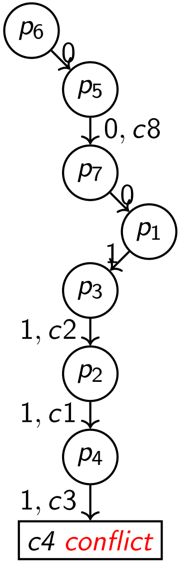
We also annotate each node with decision level.

Commentary: DAG = directed acyclic graph. *conflict* node denotes contradiction in the run. *causeClause* definition works with the conflict node.(Why?)

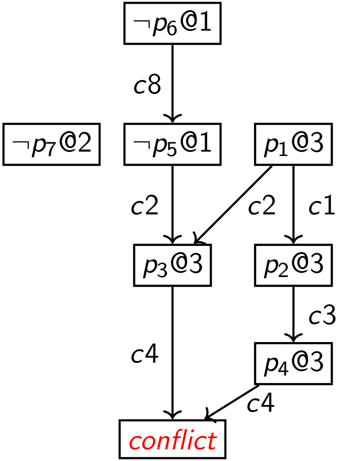
Example: implication graph

Example 6.7

$$
\begin{aligned}
c_1 &= (\neg p_1 \vee p_2)\\
c_2 &= (\neg p_1 \vee p_3 \vee p_5)\\
c_3 &= (\neg p_2 \vee p_4)\\
c_4 &= (\neg p_3 \vee \neg p_4)\\
c_5 &= (p_1 \vee p_5 \vee \neg p_2)\\
c_6 &= (p_2 \vee p_3)\\
c_7 &= (p_2 \vee \neg p_3 \vee p_7)\\
c_8 &= (p_6 \vee \neg p_5)
\end{aligned}
$$



Implication graph



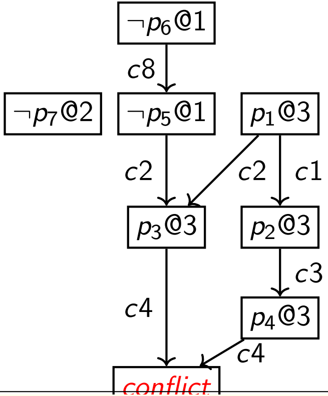
Conflict clause

We traverse the implication graph backwards to find the set of decisions that caused the conflict.

Definition 6.7

The clause of the negations of the causing decisions is called conflict clause.

Example 6.8



Conflict clause : $p_6 \vee \neg p_1$

Commentary: In the above example, p_6 is set to 0 by the first decision. Therefore, literal p_6 is added in the conflict clause. Since the second decision does not contribute to the conflict, nothing is added in the conflict clause for the decision. Since p_1 is set to 1 by the third decision, literal $\neg p_1$ is added in the conflict clause. Not an immediately obvious idea. You may need to stare at the definition for sometime.

Clause learning

Clause learning heuristics

- add conflict clause in the input clauses and
- backtrack to the second last conflicting decision, and proceed like DPLL

Theorem 6.1

Adding conflict clause

1. *does not change the set of satisfying assignments*
2. *implies that the conflicting partial assignment will never be tried again*

Example 6.9

In our running example, we added conflict clause $p_6 \lor \neg p_1$.

The second last decision in the clause is $p_6 = 0$. *We backtrack to it without flipping it.*

We run unit propagation p_1 *will be forced to be* 0 *due to the conflict clause.*

Benefit of adding conflict clauses

1. Prunes away search space
2. Records past work of the SAT solver
3. Enables many other heuristics without much complications. We will see them shortly.

Example 6.10

In the previous example, we made decisions : $m(p_6) = 0$, $m(p_7) = 0$, *and* $m(p_1) = 1$

We learned a conflict clause : $p_6 \vee \neg p_1$

Adding this clause to the input clauses results in

1. $m(p_6) = 0$, $m(p_7) = 1$, *and* $m(p_1) = 1$ *will never be tried*
2. $m(p_6) = 0$ *and* $m(p_1) = 1$ *will never occur simultaneously.*

Topic 6.3

CDCL(conflict driven clause learning)

DPLL to CDCL

Impact of clause learning was profound.

Some call the optimized algorithm CDCL(conflict driven clause learning) instead of DPLL.

Commentary: The name change is not a gimmick. All SAT solving algorithms find a satisfying assignment by attempting assignments in some sequence. An algorithm is characterized by the order of attempts. Due to the conflict clauses, CDCL exhibits very distinct probability distribution of the order of attempts from DPLL. Therefore, the change in name is warranted. Furthermore, we have very limited theoretical understanding of the success of CDCL.

CDCL as an algorithm

Algorithm 6.3: CDCL

Input: CNF F

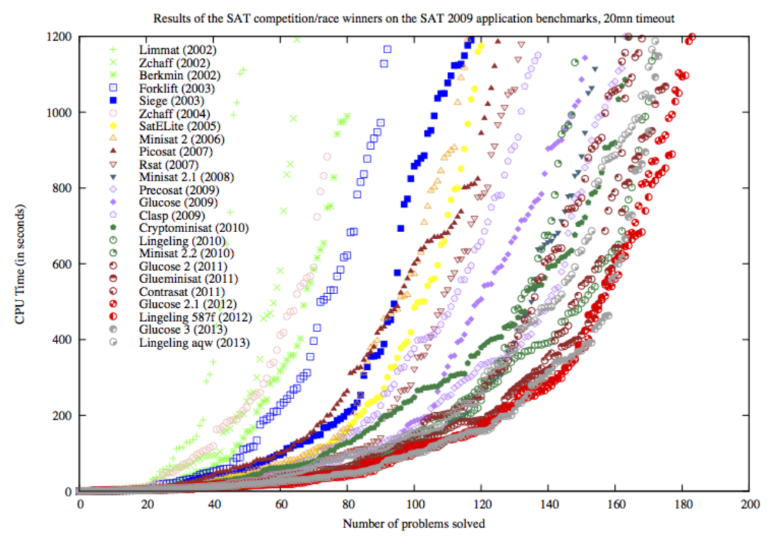
$m := \emptyset$; $dl := 0$; $dstack := \lambda x.0$;
$m :=$ UnitPropagation(m, F);
do
    // backtracking
    while F *is false under* m **do**
        if $dl = 0$ **then return** *unsat*;
        $(C, dl) :=$ AnalyzeConflict(m, F);
        $m.resize(dstack(dl))$; $F := F \cup \{C\}$;
        m := UnitPropagation(m, F);
    // Boolean decision
    if F *is unassigned under* m **then**
        $dstack(dl) := m.size()$;
        $dl := dl + 1$; $m :=$ Decide(m, F);
        m := UnitPropagation(m, F) ;
while F *is unassigned or false under* m;
return *sat*

dl stands for decision level

dstack records history of backtracking

- UnitPropagation(m, F) - applies unit propagation and extends m
- AnalyzeConflict(m, F) - returns a conflict clause learned using implication graph and a decision level upto which the solver needs to backtrack
- Decide(m, F) - chooses an unassigned variable in m and assigns a Boolean value

Efficiency of SAT solvers over the years



Source: `http://satsmt2014.forsyte.at/files/2014/07/SAT-introduction.pdf`

Cactus plot:

Y-axis: time out

X-axis: Number of problems solved

Color: a competing solver

Exercise 6.8

a. What is the negative impact of SAT competition?

b. What are look-ahead based SAT solvers?

Impact of SAT technology

Impact is enormous.

Probably, the greatest achievement of the first decade of this century in science after sequencing of human genome

A few are listed here

- Hardware verification and design assistance
  Almost all hardware/EDA companies have their own SAT solver
- Planning: many resource allocation problems are convertible to SAT
- Security: analysis of crypto algorithms
- Solving hard problems, *e. g.*, travelling salesman problem

Topic 6.4

Problems

Exercise : run CDCL

Exercise 6.9

Give a run of CDCL to completion on the CNF formula in example 6.4

Exercise: CDCL termination

Exercise 6.10

Prove that CDCL always terminates.

DPLL to Resolution*

Exercise 6.11

Let us suppose we run DPLL on an unsatisfiable formula. Give a linear time algorithm in terms of the number of steps in the run to generate resolution proof of unsatisfiability from the run of DPLL.

Lovàsz local lemma vs. SAT solvers

Here, we assume a k-CNF formula has clauses with exactly k literals.

Theorem 6.2 (Lovàsz local lemma)

If each variable in a k-CNF formula ϕ occurs less than $2^{k-2}/k$ times, ϕ is sat.

Definition 6.8

A Lovàsz formula is a k-CNF formula that has all variables occurring $\frac{2^{k-2}}{k} - 1$ times, and for each variable p, p and $\neg p$ occur nearly equal number of times.

Exercise 6.12

- *Write a program that generates uniformly random Lovàsz formula*
- *Generate 10 instances for $k = 3, 4, 5, \ldots$.*
- *Solve the instances using some sat solver*
- *Report a plot k vs. average run times*

Commentary: There are many sat solvers available online. Look into the following webpage of sat competition to find a usable and downloadable tool. `http://www.satcompetition.org`. Please discuss with the instructor if there is any confusion.

DPLL on Horn clauses (midterm 2022)

Exercise 6.13

Prove/Disprove: If we run DPLL on a set of Horn clauses, then it will never have to backtrack to check satisfiability.

End of Lecture 6