CS 433 Automated Reasoning 2025

Lecture 9: Going retro : binary decision diagram

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Retro technology

Let us go back to 90's

Commentary: SAT solving had become the focus of research by 90s. Many hard problems such as hardware verification naturally encode into SAT problem. There were several algorithms and implementations. The first break through came in early 90's.

Topic 9.1

Binary Decision Diagrams

First practical SAT solving

Binary Decision Diagram(BDD) is a data structure that enabled the first practical SAT solver.

BDDs came to prominence in early 90s.

J. R. Burch, E. M. Clarke, K. L. McMillan, D. L. Dill, and J. Hwang.
Symbolic model checking: 10^{20} states and beyond. Information and Computation, 1992.

CDCL has outsmarted BDD, but it is worth exploring.

Partial evaluation

Let us suppose a partial model m s.t. $\mathbf{Vars}(F) \not\subseteq dom(m)$.
We can assign meaning to $m(F)$, which we will denote with $F|_m$.

Definition 9.1

Let F be a formula and $m = \{p_1 \mapsto b_1, ..\}$ be a partial model.

$$\text{Let } F|_{x_i \mapsto b_i} \triangleq \begin{cases} F[\top/x_i] & \text{if } b_i = 1 \\ F[\bot/x_i] & \text{if } b_i = 0. \end{cases}$$

The partial evaluation $F|_m$ be $F|_{p_1 \mapsto b_1}|_{p_2 \mapsto b_2}| \dots$ after some simplifications.

For short hand, we may write $F|_p$ for $F|_{p \mapsto 1}$ and $F|_{\neg p}$ for $F|_{p \mapsto 0}$.

Exercise 9.1

Prove $(F|_p \wedge p) \vee (F|_{\neg p} \wedge \neg p) \equiv F$

Example : partial evaluation

Example 9.1

Consider $F = (p \vee q) \wedge r$

$$F|_p = ((p \vee q) \wedge r)[\top/p] = (\top \vee q) \wedge r \equiv \top \wedge r \equiv r$$

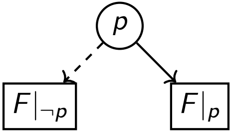
Exercise 9.2

Compute

- $((p \vee q) \wedge r)|_{\neg p}$
- $((p_1 \Leftrightarrow q_1) \wedge (p_2 \Leftrightarrow q_2))|_{p_1 \mapsto 0, p_2 \mapsto 0}$

Decision branch

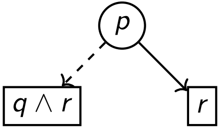
Due to the theorem in exercise 9.1, the following tree may be viewed as representing F.



Dashed arrows represent 0 decisions and solid arrows represent 1 decisions.

Example 9.2

Consider $(p \vee q) \wedge r$

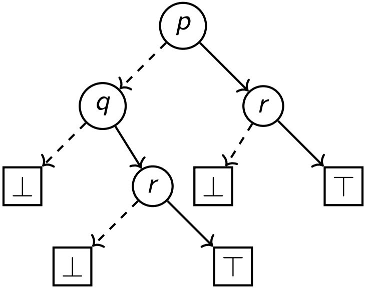


Decision tree

We may further expand $F|_{\neg p}$ and $F|_p$ until we are left with $\top$ and $\bot$ at the leaves. The obtained tree is called the decision tree for F.

Example 9.3

Consider $(p \vee q) \wedge r$



Binary decision diagram(BDD)

If two nodes represent same formula, we may rewire the incoming edges to only one of the nodes.

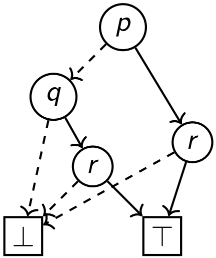
Definition 9.2

A BDD is a finite DAG such that

- *each internal node is labeled with a propositional variable*
- *each internal node has a low (dashed) and a high child (solid)*
- *there are exactly two leaves one is labelled with $\top$ and the other with $\bot$*

Example 9.4

The following is a BDD for $(p \vee q) \wedge r$



Topic 9.2

Reduced ordered binary decision diagram (ROBDD)

Optimize BDD representation

- BDD may appear an inefficient representation of formulas.
- However, we can optimize BDDs and obtain canonical representation of formulas.

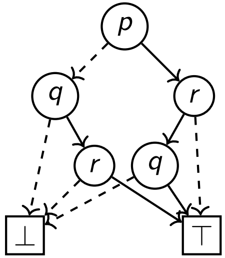
Ordered BDD (OBDD)

Definition 9.3

A BDD is ordered if there is an order $<$ over variables including $\top$ and $\bot$ such that for each node v, $v < low(v)$ and $v < high(v)$.

Example 9.5

The following BDD is not an ordered BDD



Exercise 9.3

a. Convert the above BDD into a formula

b. Give an ordered BDD of the formula

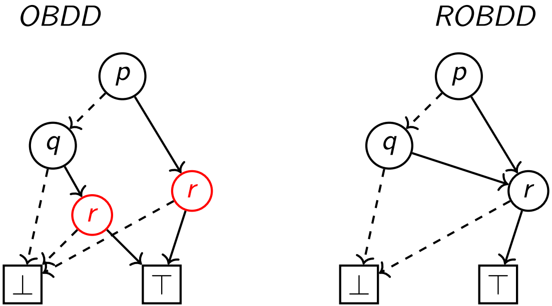
Reduced OBDD (ROBDD)

Definition 9.4

A OBDD is reduced if

- *for any nodes u and v, if $var(u) = var(v)$, $low(u) = low(v)$, $high(u) = high(v)$ then $u = v$*
- *for each node u, $low(u) \neq high(u)$*

Example 9.6



Converting to ROBDD

Any OBDD can be converted into ROBDD by iteratively applying the following transformations.

1. If there are nodes u and v such that $var(u) = var(v)$, $low(u) = low(v)$, $high(u) = high(v)$ then remove u and connect all the parents of u to v.

2. If there is a node u such that $low(u) = high(u)$ then remove u and connect all the parents of u to $low(u)$.

Exercise 9.4

Prove that the above iterations terminate.

Canonical ROBDD

Theorem 9.1

For a function $f : \mathcal{B}^n \rightarrow \mathcal{B}$ there is unique ROBDD u with ordering $p_1 < \cdots < p_n$ such that u represents $f(p_1, \dots, p_n)$.

Proof.

We use the induction over the number of parameters.

Commentary: All instance of $f() = 0$ and $f() = 1$ will use the same node $\perp$ and $\top$ respectively. We add nodes in the pool of nodes inductively.

base (n=0): There are only two functions $f() = 0$ and $f() = 1$, which are represented by nodes $\perp$ and $\top$ respectively.

step: We assume, there are unique ROBDDs for functions with less than equal to n parameters. Consider a function $f : \mathcal{B}^{n+1} \rightarrow \mathcal{B}$.

Let $f_0(p_2, \dots, p_{n+1}) \triangleq f(0, p_2, \dots, p_{n+1})$, which is represented by ROBDD u_0.
Let $f_1(p_2, \dots, p_{n+1}) \triangleq f(1, p_2, \dots, p_{n+1})$, which is represented by ROBDD u_1. ...

Canonical ROBDD (cond.) II

Proof(contd.)

case $u_0 = u_1$:

Therefore, $f = f_0 = f_1$. Therefore, u_0 represents f.

Assume there is $u' \neq u_0$ that represents f.

Therefore, $var(u') = p_1$ (Why?), $low(u') = high(u') = u_0$.

Therefore, u' is not a ROBDD. ...

Canonical ROBDD (cond.) III

Proof(contd.)

case $u_0 \neq u_1$:

Let u be such that $var(u) = p_1$, $low(u) = u_0$, and $high(u) = u_1$.

Clearly, u is a ROBDD.

Commentary: Due to the induction hypothesis, u_0 and u_1 are maximally sharing nodes, i.e., if two nodes in u_0 and u_1 represent the same function, then they must be the same. There is no further need of merger of nodes, when constructing u.

Assume there is $u' \neq u$ that represents f. Therefore, $var(u') = p_{1\text{(Why?)}}$.

Due to induction hypothesis, $low(u') = u_0$, and $high(u') = u_1$.

Due to the reduced property, $u = u'$. □

Exercise

Exercise 9.5

a. How many nodes are there in a ROBDD of an unsatisfiable formula?

a. How many nodes are there in a ROBDD of a valid formula?

Satisfiablility via BDD

Build a ROBDD that represents F.

- An unsat formula have only one node $\perp$.

Benefits of ROBDD

- If intermediate ROBDDs are small then the satisfiability check will be efficient.
- Cost of computing ROBDDs vs sizes of BDDs
- Due to the canonicity property, ROBDD is used as a formula store
- Various operations on the ROBDDs are conducive to implementation

Issues with ROBDD

- BDDs are very sensitive to the variable ordering. There are formulas that have exponential size ROBDDs for some orderings
- There is no efficient way to detect good variable orderings

Exercise 9.6

Draw the ROBDD for

$$(x_1 \wedge x_2) \vee (x_3 \wedge x_4)$$

with the following ordering on variables $x_1 < x_3 < x_2 < x_4$.

Topic 9.3

Algorithms for BDDs

Algorithms for BDDs

Next we will present algorithms for BDDs to illustrate the convenience of the data structure.

Global data structures

The algorithms maintain the following two global data structures.

$$store = (Nodes, low, high, var) := (\{\bot, \top\}, \lambda x.\texttt{null}, \lambda x.\texttt{null}, \lambda x.\texttt{null})$$

$$reverseMap : (\mathbf{Vars} \times Nodes \times Nodes) \to Nodes := \lambda x.\texttt{null}$$

Commentary: $\lambda x.\texttt{null}$ stands for a map that takes any input and returns `null`. This notation is borrowed from λ calculus.

Constructing a BDD node

Algorithm 9.1: $\textsc{MakeNode}(p, u_0, u_1)$

Input: $p \in \textbf{Vars}$, $u_0, u_1 \in \mathit{Nodes}$

if $u_0 = u_1$ **then**
  return u_0

if $\mathit{reverseMap.exists}(p, u_0, u_1)$ **then**
  return $\mathit{reverseMap.lookup}(p, u_0, u_1)$

$u := \mathit{store.add}(p, u_0, u_1)$; // allocates a new node
$\mathit{reverseMap.add}((p, u_0, u_1), u)$;
return u

Constructing BDDs from a formula

Algorithm 9.2: $\textsc{BuildROBDD}(F, p_1 < \cdots < p_n)$

Input: $F(p_1, \ldots, p_n) \in \mathbf{P}$, $p_1 < \cdots < p_n$: an ordering over variables of F

if $n = 0$ **then**

  if $F \equiv \bot$ **then return** $\bot$; **else return** $\top$;

$u_0 := \textsc{BuildROBDD}(F|_{\neg p_1}, p_2 < \cdots < p_n)$;

$u_1 := \textsc{BuildROBDD}(F|_{p_1}, p_2 < \cdots < p_n)$;

return $\textsc{MakeNode}(p_1, u_0, u_1)$

Conjunction of BDDs

Algorithm 9.3: $\textsc{ConjBDDs}(u, v)$

Input: ROBDDs u and v with same variable ordering

if $u = \bot$ *or* $v = \top$ **then return** u;

if $u = \top$ *or* $v = \bot$ **then return** v;

$u_0 := low(u); u_1 := high(u); p_u := var(u)$;

$v_0 := low(v); v_1 := high(v); p_v := var(v)$;

if $p_u = p_v$ **then**

  return $\textsc{MakeNode}(p_u, \textsc{ConjBDDs}(u_0, v_0), \textsc{ConjBDDs}(u_1, v_1))$

if $p_u < p_v$ **then**

  return $\textsc{MakeNode}(p_u, \textsc{ConjBDDs}(u_0, v), \textsc{ConjBDDs}(u_1, v))$

if $p_u > p_v$ **then**

  return $\textsc{MakeNode}(p_u, \textsc{ConjBDDs}(u, v_0), \textsc{ConjBDDs}(u, v_1))$

Exercise 9.7

Give an algorithm for computing disjunction of BDDs/not of a BDD.

Exercise: run $\textsc{ConjBDDs}$

Exercise 9.8

Consider order of variables $p_1 < p_2$. a. Draw ROBDD for $p_1 \wedge p_2$. Let us call the BDD u.
b. Draw ROBDD for $\neg p_1$. Let us call the BDD v.
c. Run $\textsc{ConjBDDs}(u, v)$

Restriction on a value

Algorithm 9.4: $\text{RESTRICT}(u, p, b)$

Input: ROBDD u with same variable ordering, variable p, $b \in \mathcal{B}$

$u_0 := low(u); u_1 := high(u); p_u := var(u);$

if $p_u = p$ *and* $b = 0$ **then**
  return $\text{RESTRICT}(u_0, p, b)$

if $p_u = p$ *and* $b = 1$ **then**
  return $\text{RESTRICT}(u_1, p, b)$

if $p_u < p$ **then**
  return $\text{MAKENODE}(p_u, \text{RESTRICT}(u_0, p, b), \text{RESTRICT}(u_1, p, b))$

if $p_u > p$ **then**
  return u

Impact of BDDs

- In 90s, BDDs revolutionized hardware verification
- Later other methods were found that are much faster and the fall of BDD was marked by the following paper,

  A. Biere,A. Cimatti,E. Clarke,Y. Zhu,
  Symbolic Model Checking without BDDs, TACAS 1999

- However, BDDs are still the heart of various software packages

Commentary: Maybe the methods that dominated the scene depend on the available computing power. The discoveries may have been predetermined. Once we reached computation power of 90's, we had BDDs. When we reached the computation power of 2000's, we had CDCL and deep learning. Maybe when we will add a few more zeros in our computing power, we may have entirely different methods that will dominate the computing scene.

Problems with BDDs

- Doing more than finding a satisfiable solution
- Variable ordering is rigid

Topic 9.4

Problems

ROBDDs

Exercise 9.9

Construct ROBDD of the following formula for the order $p < q < r < s$.

$$F = (p \vee (q \oplus r) \vee (p \vee s))$$

Let u be the ROBDD node that represents F. Give the output of $\textsc{Restrict}(u_F, p, b)$

Variable reordering

Exercise 9.10

Let u be an ROBDD with variable ordering $p_1 < ... < p_n$. Give an algorithm for transforming u into a ROBDD with ordering $p_1 < .. < p_{i-1} < p_{i+1} < p_i < p_{i+2} < .. < p_n$.

BDD-XOR

Exercise 9.11

Write an algorithm for computing xor of BDDs

BDD encoding

Exercise 9.12

Consider a and b be 2 bit wide bit-vectors. Write BDD of each of three output bits in bit-vector addition $a + b$.

BDD model counting

Exercise 9.13

a. Give an algorithm for counting models for a given ROBDD.

b. Does this algorithm work for any BDD?

End of Lecture 9