CS 433 Automated Reasoning 2025

Lecture 12: Satisfiability modulo theory (SMT) solvers

Instructor: Ashutosh Gupta

IITB India

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CDCL($\mathcal{T}$)

CDCL solves(i.e. checks satisfiability) quantifier-free propositional formulas

CDCL($\mathcal{T}$) solves quantifier-free formulas in theory $\mathcal{T}$,

- separates the boolean and theory reasoning,
- proceeds like CDCL, and
- needs support of a $\mathcal{T}$-solver $DP_{\mathcal{T}}$, i.e., a decision procedure for conjunction of literals of $\mathcal{T}$

The tools that are build using CDCL($\mathcal{T}$) are called satisfiablity modulo theory solvers (SMT solvers)

CDCL($\mathcal{T}$) - some notation

Let $\mathcal{T}$ be a first-order-logic theory with signature **S**.

We assume input formulas are from $\mathcal{T}$, quantifier-free, and in CNF.

Definition 12.1

For a quantifier-free $\mathcal{T}$ formula F, let $atoms(F)$ denote the set of atoms appearing in F.

Example 12.1

- $f(x) = g(h(x, y))$ *is a formula in QF_EUF.*
- $x > 0 \vee y + x = 3.5z$ *is a formula in QF_LRA.*

Boolean encoder

For a formula F, let boolean encoder e be a partial map from $atoms(F)$ to fresh boolean variables.

Definition 12.2

For a formula F, let $e(F)$ denote the term obtained by replacing each atom a by $e(a)$ if $e(a)$ is defined.

Example 12.2

Let $F = x < 2 \vee (y > 0 \vee x \geq 2)$ *and* $e = \{x < 2 \mapsto x_1, y > 0 \mapsto x_2\}$.
$e(F) = x_1 \vee (x_2 \vee \neg x_1)$

Exercise 12.1

Consider boolean encoder $e = \{x < 2 \mapsto x_1, y > 0 \mapsto x_2\}$. *Encode the following.*

- $e(x \geq 2) =$
- $e(x < 2 \Rightarrow y \leq 0) =$
- $e(x + y \leq 0) =$
- $e(\top) =$

Partial model

Definition 12.3

For a boolean encoder e, a partial model m is an ordered partial map from $range(e)$ to $\mathcal{B}$.

Example 12.3

partial models $\{x \mapsto 0, y \mapsto 1\}$ and $\{y \mapsto 1, x \mapsto 0\}$ are not same.

CDCL($\mathcal{T}$) will proceed by constructing partial models like CDCL.

Reverse encoder

Definition 12.4

For a partial model m of e, let $e^{-1}(m) \triangleq \{e^{-1}(x) | x \mapsto 1 \in m\} \cup \{\neg e^{-1}(x) | x \mapsto 0 \in m\}$

Example 12.4

Let $e = \{x < 2 \mapsto x_1, y > 0 \mapsto x_2\}$ *and* $m = \{x_1 \mapsto 0, x_2 \mapsto 1\}$.

$e^{-1} = \{x_1 \mapsto x < 2, x_2 \mapsto y > 0\}$

$e^{-1}(m) = \{\neg(x < 2), y > 0\}$

Exercise 12.2

Consider boolean encoder $e = \{x < 2 \mapsto x_1, y > 0 \mapsto x_2\}$. *Encode the following.*

- $e^{-1}(\{x_1 \mapsto 0\}) =$
- $e^{-1}(\{x_1 \mapsto 0, x_2 \mapsto 0\}) =$
- $e^{-1}(\{x_3 \mapsto 0\}) =$
- $e^{-1}(\emptyset) =$

Theory propagation

If we have partial assignment m, then we need to check if the theory accepts the assignment.

In other words, we need to know if $\bigwedge e^{-1}(m)$ is sat.

Example 12.5

In last example, we had $e^{-1}(m) = \{\neg(x < 2), y > 0\}$.
We ask if $\bigwedge e^{-1}(m) = \neg(x < 2) \wedge y > 0$ is sat. If no, we need to backtrack the assignments.

We assume that function $\textsc{TheoryDeduction}$ can check satisfiability of $\bigwedge e^{-1}(m)$.

CDCL($\mathcal{T}$)

Algorithm 12.1: CDCL($\mathcal{T}$)(formula G)

```
e := CreateEncoder(G); F := e(G); m := UnitPropagation(m, F); dl := 0; dstack := λx.0;
do
    // backtracking
    while m ⊭ F do
        if dl = 0 then return unsat;
        (C, dl) := AnalyzeConflict(m);
        m.resize(dstack(dl)); F := F ∪ {C}; m := UnitPropagation(m, F);
    // Boolean decision
    if F is unassigned under m then
        dstack(dl) := m.size(); dl := dl + 1; m := Decide(m, F); m := UnitPropagation(m, F) ;
    // Theory propagation
    if F is unassigned or sat under m then
        (Cs, dl') := TheoryDeduction(T)(⋀ e^{-1}(m), m, dstack, dl) ;    // Theory solving
        if dl' < dl then {dl = dl'; m.resize(dstack(dl)); } ;
        F := F ∪ e(Cs); m := UnitPropagation(m, F);
while F is unassigned under m or m ⊭ F or e^{-1}(m) is unsat;
return sat
```

F is Boolean encoding of input G

Same as SAT solver CDCL

returns a clause set and a decision level

Topic 12.1

TheoryDeduction

Theory propagation

TheoryDeduction looks at the atoms assigned so far and checks

- if they are mutually unsatisfiable
- if not, are there other literals from G that are implied by the current assignment

Any implementation must comply with the following goals

- Correctness: boolean model is consistent with $\mathcal{T}$
- Termination: unsat partial models are never repeated

TheoryDeduction

TheoryDeduction solves conjunction of literals and returns a set of clauses and a decision level.

$$(Cs, dl') := \textsc{TheoryDeduction}(\mathcal{T})(\bigwedge e^{-1}(m), m, dstack, dl)$$

Cs may contain the clauses of the form

$$(\bigwedge L) \Rightarrow \ell$$

where $\ell \in lits(F') \cup \{\bot\}$ and $L \subseteq e^{-1}(m)$.

Commentary: The RHS need not be a single literal. However, in most theories the single literal is a good practical choice.

Example : TheoryDeduction

Example 12.6

If TheoryDeduction$(QF_LRA)(x > 1 \land x < 0, ...)$ *is called, the returned clauses will be*

$$Cs := \{(x > 1 \land x < 0 \Rightarrow \bot)\}.$$

If TheoryDeduction$(QF_LRA)(x > 1 \land y > 0, ...)$ *is called, the returned clauses may be*

$$Cs := \{(x > 1 \land y > 0 \Rightarrow x + y > 0), ...\}.$$

Assuming $x + y > 0$ occurs in input

Specification of TheoryDeduction

The output of TheoryDeduction must satisfy the following conditions

- If $\bigwedge e^{-1}(m)$ is unsat in $\mathcal{T}$ then Cs must contain a clause with $\ell = \bot$. dl' is the decision level immediately after which the unsatisfiablity occurred (clearly stated shortly).
- if $\bigwedge e^{-1}(m)$ is sat then $dl' = dl$.

Example : CDCL(QF_EUF)

Example 12.7

Consider $F' = (x = y \lor y = z) \land (y \neq z \lor z = u) \land (z = x)$
$e(F') = (x_1 \lor x_2) \land (\neg x_2 \lor x_3) \land x_4$

After $F := e(F'); m := \textsc{UnitPropagation}(m, F)$
$m = \{x_4 \mapsto 1\}$

After $m := \textsc{Decide}(m, F);$
$m = \{x_4 \mapsto 1, x_2 \mapsto 0\}$

After $m := \textsc{UnitPropagation}(m, F)$
$m = \{x_4 \mapsto 1, x_2 \mapsto 0, x_1 \mapsto 1\}$

Example : CDCL(QF_EUF) II

Since $m = \{x_4 \mapsto 1, x_2 \mapsto 0, x_1 \mapsto 1\}$, $e^{-1}(m) = \{x = y, y \neq z, z = x\}$

After $(Cs, dl') :=$ $\textsc{TheoryDeduction}$(QF_EUF)$(x = y \wedge y \neq z \wedge z = x, ..)$
$Cs = \{x \neq y \vee y = z \vee z \neq x\}, dl' = 0, e(Cs) = \{\neg x_1 \vee x_2 \vee \neg x_4\}$

After $F := F \cup e(Cs); m := \textsc{UnitPropagation}()$
$m = \{x_4 \mapsto 1, x_2 \mapsto 0, x_1 \mapsto 1\}$ $\leftarrow$ conflict with learned clause

Exercise 12.3

Complete the run

Theory propagation implementation - incremental solver

Theory propagation is implemented using incremental theory solvers.

Incremental solver $DP_{\mathcal{T}}$ for theory $\mathcal{T}$

- takes input constraints as a sequence of literals,
- has a data structure that defines the solver state and satisfiability of constraints seen so far.

Theory solver $DP_{\mathcal{T}}$ interface

A theory solver must provide the following interface.

- push(ℓ) - adds literal ℓ in "constraint store"
- pop() - removes last pushed literal from the store
- checkSat() - checks satisfiability of current store
- unsatCore() - returns the set of literals that caused unsatisfiablity

Definition 12.5

An unsat core of Σ is a subset (preferably minimal) of Σ that is unsat.

Commentary: We assume that push and pop call checkSat() at the end of their execution. Therefore, explicit calls to checkSat() are not necessary. However, practical tools allow users to choose the policy of calling checkSat() - lazy vs. eager

Theory propagation implementation

Algorithm 12.2: THEORYDEDUCTION

Input: Set of literals Ls

Read only input: m partial model, $dstack$ decision depths, dl current decision level, input formula G

```
foreach ℓ ∈ Ls do
    DP_T.push(ℓ)
if DP_T.checkSat() == unsat then
    // theory conflict
    Ls' := DP_T.unsatCore(); dl' := max{dl''|∃ℓ ∈ Ls', i. m[i] = e(ℓ) ∧ dstack(dl'') < i};
    return ({¬⋀Ls'}, dl')
else
    //implied clauses
    Cs := ∅;
    foreach ℓ ∈ Lits(G) do
        DP_T.push(¬ℓ);
        if DP_T.checkSat() == unsat then
            Ls' := DP_T.unsatCore(); Cs := Cs ∪ {¬⋀Ls'};
        DP_T.pop();
    return (Cs,dl)
```

dl' is the latest decision after which all literals in Ls' became true.

$Ls' = Ls$ will also be correct. But, inefficient.

ℓ is called implied literal and $\neg\ell \in Ls'$

Example: Theory deduction unsat example

Example 12.8

Consider $Ls = \{x = z, x = y, f(x) \neq f(y)\}$

First we will push all the literals to the theory solver.
$DP_{\mathcal{T}}.push(x = z); DP_{\mathcal{T}}.push(x = y); DP_{\mathcal{T}}.push(f(x) \neq f(y))$.

We will call $DP_{\mathcal{T}}.checkSat()$, *which will return unsat.*

We will call $DP_{\mathcal{T}}.unsatCore()$, *which will return* $\{x = y, f(x) \neq f(y)\}$.

The returned clause will be $x \neq y \vee f(x) = f(y)$.

Theory deduction will also return an appropriate decision level.

Example: Theory deduction sat example

Example 12.9

Consider $x = y \in Ls$ *and assume* $f(x) = f(y) \in Lits(G)$.

After pushing Ls, *let us assume* $DP_{\mathcal{T}}.checkSat()$ *returns sat.*

We search for implied clauses.

Since $f(x) = f(y) \in Lits(G)$, *we will eventually call* $DP_{\mathcal{T}}.push(f(x) \neq f(y))$.

We get unsatisfiablity and unsat core, $\{x = y, f(x) \neq f(y)\}$.

We return $x \neq y \lor f(x) = f(y)$ *among the implied clauses.*

Topic 12.2

Example theory propagation implementation

Let us study implementation of DP_{EUF}

DP_{EUF} decides conjunction of literals in the theory of EUF with interface

push, pop, checkSat, and unsatCore.

push, checkSat, and pop

- $DP_{EUF}.push$

Algorithm 12.3: $DP_{EUF}.push(t_1 \bowtie t_2)$

```
IncrEUF(t1 ⋈ t2);
```

- $DP_{EUF}.checkSat()$ { **return** $conflictFound$; }

- $DP_{EUF}.pop()$ is implemented by recording the time stamp of pushes and undoing all the mergers happened after the last push.

Exercise 12.4

Write pseudo code for $DP_{EUF}.pop()$

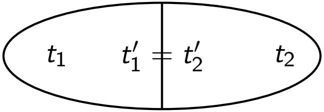
Unsat core

Algorithm 12.4: $DP_{EUF}.unsatCore()$

assume($conflictFound = 1$);
Let $(t_1 \neq t_2)$ be the disequality that was violated;
return $\{t_1 \neq t_2\} \cup getReason(t_1, t_2)$;

Algorithm 12.5: $getReason(t_1, t_2)$

Let $(t'_1 = t'_2)$ be the merge operation that placed t_1 and t_2 in same class;
if $t'_1 = f(s_1, .., s_k) = f(u_1, ...u_k) = t'_2$ *was derived due to congruence* **then**
    $reason := \bigcup_i getReason(s_i, u_i)$
else
    $reason := \{t'_1 = t'_2\}$
return $getReason(t_1, t'_1) \cup reason \cup getReason(t'_2, t_2)$



Example: unsat core

Example 12.10

Consider unsatisfiable constraints: $x = z \wedge y = z \wedge f(x) \neq f(y)$

There must be *exactly* one disequality involved in the contradiction.

$$f(x) \neq f(y)$$

Therefore, we look for the reason for $f(x) = f(y)$.

Since $f(x)$ and $f(y)$ was made equal due to congruence, we look for reasons for $x = y$.

Since x and y joined the same class during the processing of input inequality $y = z$, it is part of the unsat core.

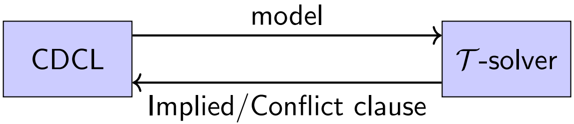
Now we need to find the reason of $x = z$, which is one of our input.

Topic 12.3

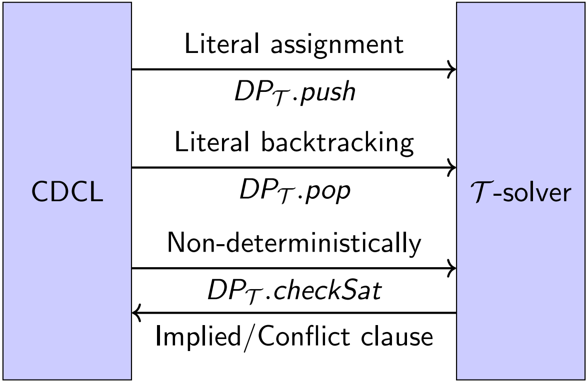
SMT Solvers

Incremantal theory propagation

Earlier $CDCL(\mathcal{T})$



Fine-grained interaction with theory



Theory propagation strategies

- Exhaustive or Eager :
  Cs contains all possible clauses
- Minimal or Lazy :
  Cs only contains the clause that refutes current m
- Somewhat Lazy :
  Cs contains only easy to deduce clauses

Rise of SMT solvers

- In early 2000s, stable SMT solvers started appearing. e.g., Yiecs
- SMT competition(SMT-comp) became a driving force in their ever increasing efficiency
- Formal methods community quickly realized their potential
- Z3, one of the leading SMT solver, alone has about 3000+ citations (375 per year)(June 2016)

Leading tools

The following are some of the leading SMT solvers

- Z3
- CVC4
- MathSAT
- Boolector

Topic 12.4

Problems

Run SMT solvers

Exercise 12.5

- *Find a satisfying assignment of the following formula using SMT solver*

$$(x > 0 \vee y < 0) \wedge (x + y > 0 \vee x - y < 0)$$

  Give the model generated by the SMT solver.

- *Prove the following formula is valid using SMT solver*

$$(x > y \wedge y > z) \Rightarrow x > z$$

  Give the proof generated by the SMT solver.

Please do not simply submit the output. Please write the answers in the mathematical notation.

Knapsack problem

Exercise 12.6

Write a program for solving the knapsack problem that requires filling a knapsack with stuff with maximum value. For more information look at the following.

`https://en.wikipedia.org/wiki/Knapsack_problem`

The output of the program should be the number of solutions that have value more than 95% of the best value.

Download Z3 from the following webpage: https://github.com/Z3Prover/z3

We need a tool to feed random inputs to your tool. Write a tool that generates random instances, similar to what was provided last time.

Evaluate the performance on reasonably sized problems. You also need to design the evaluation strategy. Evaluation plots and a small text to describe your strategies.

Topic 12.5

Extra slides : optimizations

Implied literals without implied clauses

Bottleneck: There may be too many implied clauses.

Observation: Very few of the implied clauses are useful, i.e., contribute in early detection of conflict.

Optimization: apply implied literals, without adding implied clauses.

Optimization overhead: If an implied model is used in conflict then recompute the implied clause for the implication graph analysis.

Relevancy

Bottleneck: All the assigned literals are sent to the theory solver.

Observation: However, *CDCL* only needs to send those literals to the solver that make unique clauses satisfiable.

Optimization:

- Each clause chooses one literal that makes it sat under current model.
- Those clause that are not sat under current model do nothing.
- If a literal is not chosen by any clause then it is not passed on to $\mathcal{T}$-solver.

Patented: US8140459 by Z3 guys(the original idea is more general than stated here)

Optimization overhead: Relevant literal management

Exercise 12.7

Suggest a scheme for relevant literal management.

End of Lecture 12