

CS 433 Automated Reasoning 2025

Lecture 20: Theory combination

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Theory combination

A formula may have terms that involved multiple theories.

Example 20.1

$$\neg P(y) \wedge s = \text{store}(t, i, 0) \wedge x - y - z = 0 \wedge z + s[i] = f(x - y) \wedge P(x - f(f(z)))$$

The above formula involves theory of

- ▶ equality \mathcal{T}_E
- ▶ linear integer arithmetic \mathcal{T}_Z
- ▶ arrays \mathcal{T}_A

How to check satisfiability of the formula?

Combination solving

Let suppose a formula refers to theories $\mathcal{T}_1, \dots, \mathcal{T}_k$.

We will assume that we have decision procedures for each quantifier-free \mathcal{T}_i .

We will present a method **that combines the decision procedures** and provides a decision procedure for quantifier-free $Cn(\mathcal{T}_1 \cup \dots \cup \mathcal{T}_k)$.

Topic 20.1

Nelson-Oppen method

Nelson-Oppen method conditions

The Nelson-Oppen method combines theories that satisfy the following conditions

1. The signatures \mathbf{S}_i are disjoint.
2. The theories are **stably infinite**
3. The formulas are conjunction of quantifier-free literals

Stably infinite theories

Definition 20.1

A theory is **stably infinite** if each quantifier-free satisfiable formula under the theory is satisfiable in an infinite model.

Example 20.2

Let us suppose we have the following axiom in a theory

$$\forall x, y, z. (x = y \vee y = z \vee z = x)$$

The above formula says that there are **at most two elements** in the domain of a satisfying model. Therefore, the theory is **not stably infinite**.

Nelson-Oppen method terminology I

We call a function/predicate in \mathbf{S}_i an *i*-symbol.

Definition 20.2

A *term* t is an *i*-term if the top symbol is an *i*-symbol.

Definition 20.3

An *i*-atom is

- ▶ an *i*-predicate atom,
- ▶ $s = t$, where s is an *i*-term, or
- ▶ $v = t$, v is a variable and t is an *i*-term.

Definition 20.4

An *i*-literal is an *i*-atom or the negation of one.

Exercise 20.1

Let \mathcal{T}_E , \mathcal{T}_Z , and \mathcal{T}_A be involved in a formula.

- ▶ $x + y$ is
- ▶ $store(A, x, f(x + y))$ is
- ▶ $A[3] \leq f(x)$ is
- ▶ $f(x) = 3 + y$ is
- ▶ $z = 3 + y$ is
- ▶ $z \neq 3 + y$ is

Nelson-Oppen method terminology II

Definition 20.5

An occurrence of a term t in i -term/literal is **i -alien** if t is a j -term for $i \neq j$ and all of its superterms are i -terms.

Definition 20.6

An expression is **pure** if it contains only variables and i -symbols for some i .

Exercise 20.2

Let \mathcal{T}_E , \mathcal{T}_Z , and \mathcal{T}_A be involved in a formula. Find the alien term.

► In $A[3] = f(x)$,

► In $z = 3 + y$,

► In $f(x) \neq f(2)$,

► In $f(x) = A[3]$,

► In $store(a, x + y, f(z))$,

Nelson-Oppen method: convert to separate form

Let F be a conjunction of literals.

We produce an equiv-satisfiable $F_1 \wedge \dots \wedge F_k$ such that F_i is a \mathcal{T}_i formula.

1. Pick an i -literal $\ell \in F$ for some i . $F := F - \{\ell\}$.
2. If ℓ is pure, $F_i := F_i \cup \{\ell\}$.
3. Otherwise, there is a term t occurring i -alien in ℓ .
Let z be a fresh variable. $F := F \cup \{\ell[t \mapsto z], z = t\}$.
4. go to step 1.

Example 20.3

Consider $1 \leq x \leq 2 \wedge f(x) \neq f(2) \wedge f(x) \neq f(1)$ of theory $Cn(\mathcal{T}_E \cup \mathcal{T}_Z)$.

Alien terms are $\{2, 1\}$.

In separate form, $F_E = f(x) \neq f(z) \wedge f(x) \neq f(y)$ $F_Z = 1 \leq x \leq 2 \wedge y = 1 \wedge z = 2$

Theory solvers need to coordinate

Let DP_i be the decision procedure of theory \mathcal{T}_i .

F is unsatisfiable if for some i , $DP_i(F_i)$ returns unsatisfiable.

However, if all $DP_i(F_i)$ return satisfiable, we **can not guarantee** satisfiability.

The decision procedures **need to coordinate** to check the satisfiability.

Equivalence constraints

Definition 20.7

Let S be a set of terms and equivalence relation \sim over S .

$$F[\sim] := \bigwedge \{t = s \mid t \sim s \text{ and } t, s \in S\} \wedge \bigwedge \{t \neq s \mid t \not\sim s \text{ and } t, s \in S\}$$

$F[\sim]$ will be used for the coordination.

Non-deterministic Nelson-Oppen method

Let \mathcal{T}_1 and \mathcal{T}_2 be two theories with disjoint signature.

Let F be a conjunction of literals for theory $Cn(\mathcal{T}_1 \cup \mathcal{T}_2)$.

1. Convert F to separate form $F_1 \wedge F_2$.
2. **Guess** an equivalence relation \sim over variables $vars(F_1) \cap vars(F_2)$.
3. Run $DP_1(F_1 \wedge F[\sim])$
4. Run $DP_2(F_2 \wedge F[\sim])$

If there is a \sim such that both steps 3 and 4 return satisfiable, F is satisfiable.

Otherwise F is unsatisfiable.

Exercise 20.3

Extend the above method for k theories.

Example: non-deterministic Nelson-Oppen method

Example 20.4

We had the following formula in separate form.

$$F_E = f(x) \neq f(z) \wedge f(x) \neq f(y) \qquad F_Z = 1 \leq x \leq 2 \wedge y = 1 \wedge z = 2$$

Common variables x , y , and z .

Five potential $F[\sim]$ s

1. $x = y \wedge y = z \wedge z = x$: Inconsistent with F_E
2. $x = y \wedge y \neq z \wedge z \neq x$: Inconsistent with F_E
3. $x \neq y \wedge y \neq z \wedge z = x$: Inconsistent with F_E
4. $x \neq y \wedge y = z \wedge z \neq x$: Inconsistent with F_Z
5. $x \neq y \wedge y \neq z \wedge z \neq x$: Inconsistent with F_Z

Since all \sim are causing inconsistency, the formula is unsatisfiable.

Topic 20.2

Correctness of Nelson-Oppen

model and assignment

We have noticed if there are no quantifiers, **variables behave like constants**.

In the lecture, we will refer models and assignments together as models.

Definition 20.8

Let m be a model of signature \mathbf{S} and variables V . Let $m|_{\mathbf{S}', V}$ be the restriction of m to the symbols in \mathbf{S}' and the variables in V .

Homomorphisms and isomorphism of models

Definition 20.9

Consider signature $\mathbf{S} = (\mathbf{F}, \mathbf{R})$ and a variables V . Let m and m' be \mathbf{S} , V -models. A function $h : D_m \rightarrow D_{m'}$ is a **homomorphism** of m into m' if the following holds.

- ▶ for each $f/n \in \mathbf{F}$ and $(d_1, \dots, d_n) \in D_m^n$, $h(f_m(d_1, \dots, d_n)) = f_{m'}(h(d_1), \dots, h(d_n))$
- ▶ for each $P/n \in \mathbf{R}$ and $(d_1, \dots, d_n) \in D_m^n$, $(d_1, \dots, d_n) \in P_m$ iff $(h(d_1), \dots, h(d_n)) \in P_{m'}$
- ▶ for each $v \in V$, $h(v_m) = v_{m'}$

Definition 20.10

A homomorphism h of m into m' is called **isomorphism** if h is one-to-one. m and m' are called **isomorphic** if an h exists that is also onto.

Isomorphic models ensure combined satisfiability

Theorem 20.1

Let F_i be a \mathbf{S}_i -formula with variables V_i for $i \in \{1, 2\}$. $F_1 \wedge F_2$ is satisfiable iff there are $m_1 \models F_1$ and $m_2 \models F_2$ such that

$$m_1|_{\mathbf{s}_1 \cap \mathbf{s}_2, V_1 \cap V_2} \text{ is isomorphic to } m_2|_{\mathbf{s}_1 \cap \mathbf{s}_2, V_1 \cap V_2}.$$

Proof.

(\Rightarrow) trivial. (Why?)

(\Leftarrow).

We have models $m_1 \models F_1$ and $m_2 \models F_2$.

Let h be the onto isomorphism from $m_1|_{\mathbf{s}_1 \cap \mathbf{s}_2, V_1 \cap V_2}$ to $m_2|_{\mathbf{s}_1 \cap \mathbf{s}_2, V_1 \cap V_2}$.

We construct a model m for $F_1 \wedge F_2$.

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Isomorphic models ensure combined satisfiability II

Proof(contd.)

Let $D_m \triangleq D_{m_1}$ and $m|_{\mathbf{S}_1, V_1} \triangleq m_1$.

We are yet to give meaning to symbols that are not in \mathbf{S}_1 and V_1 . Let us give meaning to the rest.

- ▶ For $v \in V_2 - V_1$, $v_m \triangleq h^{-1}(v_{m_2})$
- ▶ For $f/n \in \mathbf{S}_2 - \mathbf{S}_1$, $f_m(d_1, \dots, d_n) \triangleq h^{-1}(f_{m_2}(h(d_1), \dots, h(d_n)))$
- ▶ ... similarly for predicates.

Clearly $m \models F_1$. Since $m|_{\mathbf{S}_2, V_2}$ and m_2 are isomorphic, $m \models F_2$. (Why?)

Therefore, $m \models F_1 \wedge F_2$. □

Equality preserving models ensure combined satisfiability

Theorem 20.2

Let F_i be a \mathbf{S}_i -formula with variables V_i for $i \in \{1, 2\}$. Let $\mathbf{S}_1 \cap \mathbf{S}_2 = \emptyset$. $F_1 \wedge F_2$ is satisfiable iff there are $m_1 \models F_1$ and $m_2 \models F_2$ such that

- ▶ $|D_{m_1}| = |D_{m_2}|$ and
- ▶ $x_{m_1} = y_{m_1}$ iff $x_{m_2} = y_{m_2}$ for each $x, y \in V_1 \cap V_2$

Proof.

(\Rightarrow) trivial. (Why?)

(\Leftarrow) .

Let $V_m = \{v_m \mid v \in V\}$. Let $h : (V_1 \cap V_2)_{m_1} \rightarrow (V_1 \cap V_2)_{m_2}$ be defined as follows

$$h(v_{m_1}) := v_{m_2} \quad \text{for each } v \in V_1 \cap V_2.$$

h is well-defined (Why?), one-to-one (Why?), and onto (Why?).

...

Exercise 20.4 Prove the above

Equality preserving models ensure combined satisfiability II

Proof(contd.)

Therefore, $|(V_1 \cap V_2)_{m_1}| = |(V_1 \cap V_2)_{m_2}|$

Therefore, $|D_{m_1} - (V_1 \cap V_2)_{m_1}| = |D_{m_2} - (V_1 \cap V_2)_{m_2}|$

Therefore, we can extend h to $h' : D_{m_1} \mapsto D_{m_2}$ that is one-to-one and onto. (Why?)

By construction, h' is isomorphism from $m_1|_{V_1 \cap V_2}$ to $m_2|_{V_1 \cap V_2}$.

Therefore, by the previous theorem, $F_1 \wedge F_2$ is satisfiable. □

Nelson-Oppen correctness

Theorem 20.3

Let \mathcal{T}_i be stably infinite \mathbf{S}_i -theory and F_i be \mathbf{S}_i a formula with variables V_i for $i \in \{1, 2\}$. Let $\mathbf{S}_1 \cap \mathbf{S}_2 = \emptyset$. $F_1 \wedge F_2$ is $Cn(\mathcal{T}_1 \cup \mathcal{T}_2)$ -satisfiable iff there is an equivalence relation \sim over $V_1 \cap V_2$ such that $F_i \wedge F[\sim]$ is \mathcal{T}_i -satisfiable.

Proof.

(\Rightarrow) trivial. (Why?)

(\Leftarrow). Suppose there is \sim over $V_1 \cap V_2$ such that $F_i \wedge F[\sim]$ is \mathcal{T}_i -satisfiable.

Since \mathcal{T}_i is stably infinite, there is an infinite model $m_i \models F_i \wedge F[\sim]$.

Due to LST (a standard theorem), $|m_1|$ and $|m_2|$ are infinity of same size.

Due to $m_1 \models F[\sim]$ and $m_2 \models F[\sim]$, $x_{m_1} = y_{m_1}$ iff $x_{m_2} = y_{m_2}$ for each $x, y \in V_1 \cap V_2$.

Due to the previous theorem, $F_1 \wedge F_2$ is $Cn(\mathcal{T}_1 \cup \mathcal{T}_2)$ -satisfiable. □

Topic 20.3

Implementation of Nelson-Oppen

Searching \sim

Enumerating \sim over shared variables S is very expensive.

Exercise 20.5

Let $|S| = n$. How many \sim are there?

The goal is to minimize the search.

- ▶ Reduce the size of S by simplifying formulas.
- ▶ Efficient strategy of finding \sim

Commentary: In the simplification, we replace alien terms with native terms as much as possible.

Efficient search for \sim

We can use DPLL like search for \sim .

- ▶ **Decision:** Incrementally add a (dis)equality in \sim .
- ▶ **Backtracking:** backtrack if a theory finds inconsistency and ensure early detection of inconsistency.
- ▶ **Propagation:** If an (dis)equality is implied by a current $F_i \wedge F[\sim]$ add them to \sim .

For convex theories, this strategy is **very efficient**. There is **no need** for decisions.

Convex theories

Definition 20.11

\mathcal{T} is **convex** if for a conjunction literals F and variables $x_1, \dots, x_n, y_1, \dots, y_n$, $F \Rightarrow_{\mathcal{T}} x_1 = y_1 \vee \dots \vee x_n = y_n$ implies for some $i \in 1..n$, $F \Rightarrow_{\mathcal{T}} x_i = y_i$.

Example 20.5

$\mathcal{T}_{\mathbb{Q}}$ is convex and unfortunately $\mathcal{T}_{\mathbb{Z}}$ is not convex. Consider the following implication in $\mathcal{T}_{\mathbb{Z}}$.

$$1 \leq x \leq 2 \wedge y = 1 \wedge z = 2 \Rightarrow y = x \vee z = x$$

From the above we can not conclude that the LHS implies any of the equality in RHS.

Exercise 20.6

Is the theory of arrays convex? Hint: apply axiom 2

Exercise 20.7

Prove that if all theories are convex, there is no need for decision step in the previous slide?

(Hint: Introduce disequalities between equivalence classes. Show due to convexity, F_i s will remain satisfiable.)

Incremental theory combination

Let F be a conjunctive input formula. Let S be a set of terms at the start.

1. If F is empty, return satisfiable.
2. Pick an i -literal $\ell \in F$ for some i . $F := F - \{\ell\}$.
3. Simplify and purify ℓ to ℓ' and add the fresh variable names for alien terms to S
4. $F_i := F_i \cup \{\ell'\}$.
5. If F_i is unsatisfiable, return unsatisfiable.
6. For each $s, t \in S$, check if $F_i \Rightarrow t = s$ or $F_i \Rightarrow t \neq s$, add the fact to the other F_j s.
7. go to step 1.

If theories were convex then the above algorithm returns the answer. Otherwise, we need to explore **far reduced space** for \sim in case of satisfiable response.

Example: Nelson-Oppen on convex theories == (Dis)Equality exchange

Example 20.6

Consider formula: $f(f(x) - f(y)) \neq f(z) \wedge x \leq y \wedge y + z \leq x \wedge 0 \leq z$

After separation we obtain two formulas in theory of equality and \mathbb{Q} :

$$F_E = f(w) \neq f(z) \wedge u = f(x) \wedge v = f(y) \quad F_Q = x \leq y \wedge y + z \leq x \wedge 0 \leq z \wedge u - v = w$$

Common symbols $S = \{w, u, v, z, x, y\}$.

Action	\mathcal{T}_Q	\mathcal{T}_E
Equality discovery:	$F_Q \Rightarrow x = y$	
Equality exchange and discovery:		$F_E \wedge x = y \Rightarrow u = v$
Equality exchange and discovery:	$F_Q \wedge u = v \Rightarrow w = z_{(\text{Why?})}$	
Equality exchange:		$F_E \wedge x = y \wedge w = z \Rightarrow \perp$

Contradiction. The formula is unsatisfiable.

Example: Nelson-Oppen on non-convex theories == (Dis)Equality exchange + case split

Example 20.7

Consider formula in $\mathcal{T}_E \cup \mathcal{T}_{\mathbb{Z}}$: $1 \leq x \leq 2 \wedge f(x) \neq f(1) \wedge f(x) \neq f(2)$

After separation we obtain two formulas in theory of equality and \mathbb{Z} :

$$F_E = f(x) \neq f(y) \wedge f(x) \neq f(z) \quad F_{\mathbb{Z}} = 1 \leq x \leq 2 \wedge y = 1 \wedge z = 2$$

Common symbols $S = \{x, y, z\}$.

Action	$\mathcal{T}_{\mathbb{Z}}$	\mathcal{T}_E
Disjunctive equality discovery:	$F_{\mathbb{Z}} \Rightarrow x = y \vee x = z$	
Equality case $x = y$:		$F_E \wedge x = y \Rightarrow \perp$
Equality case $x = z$:		$F_E \wedge x = z \Rightarrow \perp$

Contradiction. The formula is unsatisfiable.

Example: a satisfiable formula

Example 20.8

Consider formula in $\mathcal{T}_E \cup \mathcal{T}_{\mathbb{Z}}$: $1 \leq x \leq 3 \wedge f(x) \neq f(1) \wedge f(x) \neq f(3) \wedge f(1) \neq f(2)$

After separation we obtain two formulas in theory of equality and \mathbb{Z} :

$$F_E = f(x) \neq f(y) \wedge f(x) \neq f(w) \wedge f(y) \neq f(z) \quad F_{\mathbb{Z}} = 1 \leq x \leq 3 \wedge y = 1 \wedge z = 2 \wedge w = 3$$

Common symbols $S = \{x, y, z, w\}$.

Action	$\mathcal{T}_{\mathbb{Z}}$	\mathcal{T}_E
Equality discovery:	$F_{\mathbb{Z}} \Rightarrow x = y \vee x = z \vee x = w$ $F_{\mathbb{Z}} \Rightarrow \text{distinct}(y, z, w)$	
Equality case $x = y$:		$F_E \wedge x = y \wedge \text{distinct}(y, z, w) \Rightarrow \perp$
Equality case $x = w$:		$F_E \wedge x = w \wedge \text{distinct}(y, z, w) \Rightarrow \perp$
Equality case $x = z$:		$F_E \wedge x = z \wedge \text{distinct}(y, z, w) \not\Rightarrow \perp$

Commentary: $\text{distinct}(y, z, w) \triangleq y \neq z \wedge z \neq w \wedge w \neq y$

Topic 20.4

Problems

Theory combination

Exercise 20.8

Consider the following formula in the theory of rationals and uninterpreted functions. Apply Nelson-Oppen method to check the satisfiability of it.

$$g(a) = c + 5 \wedge f(g(a)) \geq c + 1 \wedge h(b) = d + 4 \wedge d = c + 1 \wedge f(h(b)) < c + 1$$

You need to show steps of the method. You also need to show derivation steps of the theory rules of rationals and uninterpreted functions. For strict inequalities, adjust the Comb rule accordingly.

End of Lecture 20