CS213/293 Data Structure and Algorithms 2025

Lecture 7: Red-Black Trees

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Topic 7.1

Balance and rotation

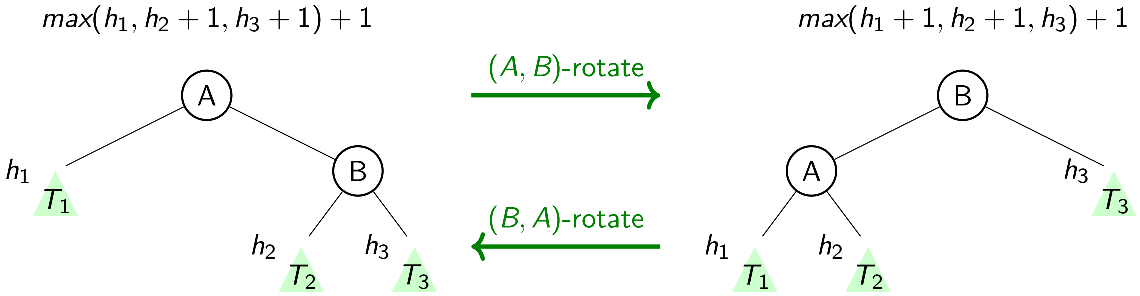
Maintain balance

BST may have a large height, which is bad for the algorithms.

Height is directly related to branching. More branching implies a shorter height.

We call BST imbalanced when the difference between the left and right subtree height is large.

Balancing height by rotation



$max(h_1, h_2 + 1, h_3 + 1) + 1$
$max(h_1 + 1, h_2 + 1, h_3) + 1$
(A, B)-rotate
(B, A)-rotate
A
B
h_1
T_1
h_2
T_2
h_3
T_3

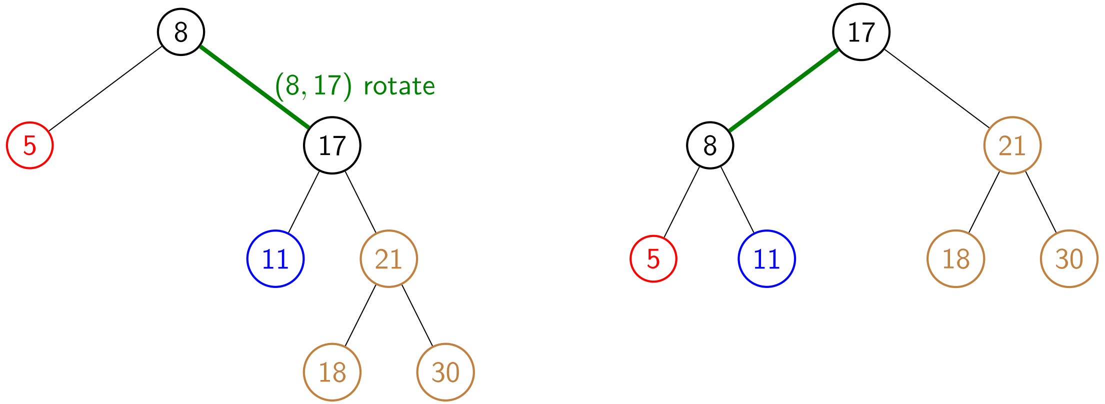
For example, if $h_3 > h_2 = h_1$ and we rotate the BST, we will get a valid and more balanced BST with less height.

Rotation may be applied in the reverse direction, where A is the left child of B. We define the symmetric rotation in both directions.

Example: rotation

Example 7.1

In the following BST, we can rotate 8-17 edge.



Commentary: This tree operation is important. Please carefully observe the destination of red,blue, and brown subtrees.

When to rotate? Can only rotation fix imbalance?

Rotation is a local operation, which must be guided by non-local measure height.

We need a definition of balance such that rotations operations should be able to achieve the balance.

Design principle:
We minimize the number of rotations while allowing some imbalance.

Topic 7.2

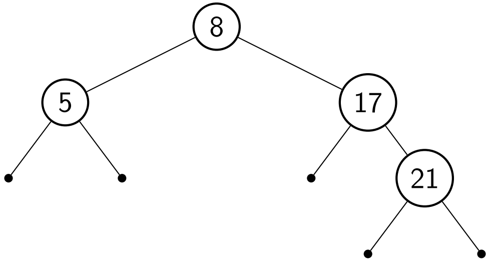
Red-black tree

Null leaves

To describe a red-black tree, we replace the null pointers for absent children with dummy null nodes.

Example 7.2

The following tiny nodes are the dummy null nodes.



Red-black tree

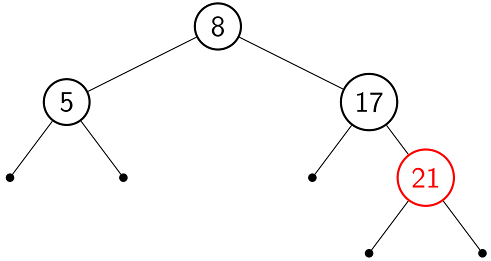
Definition 7.1

A red-black tree T is a binary search tree such that the following holds.

- All internal nodes are colored either **red** or **black**
- Null leaves have no color
- Root is colored black
- No **red** node has a **red** child
- All paths from the root to null leaves have the same number of **black** nodes.

Example 7.3

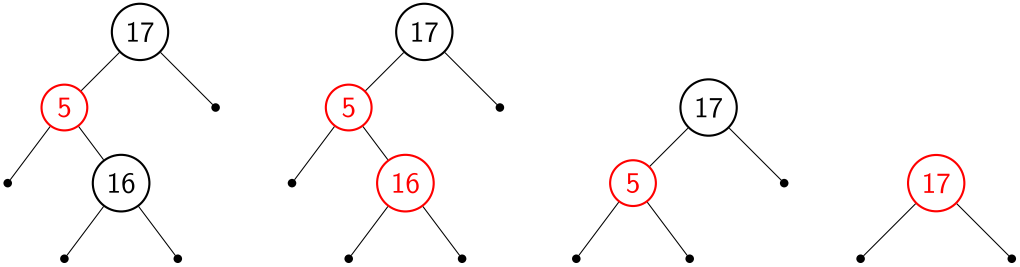
An example of a red-black tree.



Exercise: Identify red-black trees

Exercise 7.1

Which of the following are red-black trees?



Observations:

- Red nodes are not counted in the imbalance. We need them only when there is an imbalance.
- There cannot be too many red nodes. (Why?)
- Red nodes can be at every level except the root.

Black height

Definition 7.2

The black height (bh) for each node is defined as follows.

$$bh(n) = \begin{cases} 0 & \text{n is a null leaf} \\ max(bh(right(n)), bh(left(n))) + 1 & \text{n is a } \textbf{black} \text{ node} \\ max(bh(right(n)), bh(left(n))) & \text{n is a red node} \end{cases}$$

Until it is stated otherwise, all heights mentioned in this lecture are black heights.

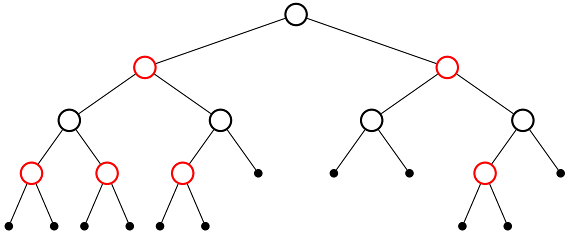
Exercise 7.2

Prove that for each node n in a red-black tree, $bh(right(n)) == bh(left(n))$.

Example: black height

Example 7.4

The black height of the following red-black tree is 2.



Exercise 7.3

Can we change the color of some nodes without breaking the conditions of a red-black tree?

Bound on the height of a red-black tree

Let h be the black height of a red-black tree containing n nodes.

- n is the smallest when all nodes are **black**. Therefore, the tree is a complete binary tree. Therefore, $n = 2^h - 1$.
- n is largest when the alternate levels of the tree are red. The height of the tree is $2h$. Therefore, $n = 2^{2h} - 1$.

$$\log_4 n < h \leq \log_2(n+1)$$

Exercise 7.4

Define red-black tree inductively and write the proof of the above bounds using induction.

Commentary: Our definition of red-black tree is not inductive therefore the above proof does not read so formal. However, an inductive definition would make proof formal. Can you write an inductive definition of red-black tree and prove the above statements?

Search, Maximum/Minimum, and Successor/Predecessor

We can search, maximum/minimum, and successor/predecessor on the red-black tree as usual.

Their running time will be $O(\log n)$ because $h < 1 + \log_2 n$.

How do we do insertion and deletion on a red-black tree?

Topic 7.3

Insertion in red-black tree*

BST insertion in red-black tree

1. Follow the usual algorithm of insertion in the BST, which inserts the new node *n* as a leaf.
   - Note that there are dummy nodes. *n* is inserted as the parent of a dummy node.

2. We color *n* red.

- Good news: No change in the black height of the tree.
- Bad news: *n* may have a *red* parent.

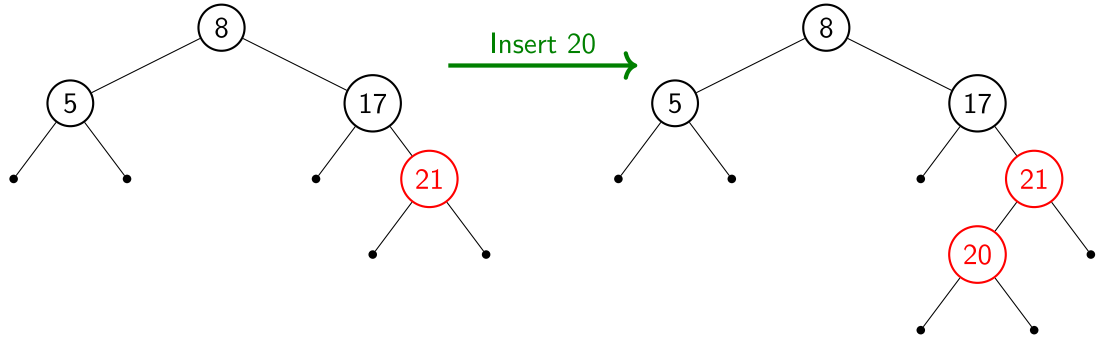
After insertion, we may have a **red-red** violation, where a **red** node has a **red** child.

Commentary: We have null nodes in this setting. We need to add nulls as children to the new node.

Example: insert in red-black tree

Example 7.5

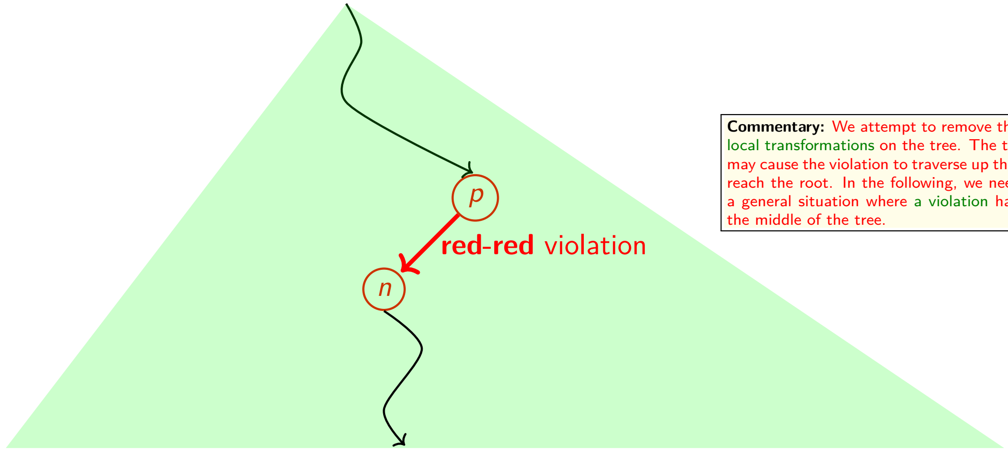
Inserting 20 in the following tree.



The insertion results in violation of the condition of the red-black tree, which says red nodes can only have black children.

Iteratively remove **red-red** violation

A **red-red** violation starts at a leaf. However, an attempt to remove the violation may push up the violation to the parent.

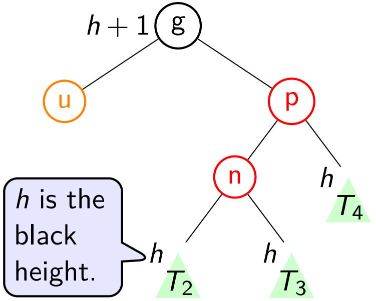


Commentary: We attempt to remove the violation by local transformations on the tree. The transformation may cause the violation to traverse up the tree until we reach the root. In the following, we need to consider a general situation where a violation has occurred in the middle of the tree.

Pattern around red-red violation

Commentary: Since the new node is always added at the leaf, one may think that h must be zero, which may not be the case. As noted in the previous slide, the violation may move upwards.

If n has a **red** parent, we correct the error by rotation and re-coloring operations.



Orange means that we need to consider all possible colors of the nodes.

We have three cases.

- Case 1: u is **red**
- Case 2: u is not **red** and the g to n path is not straight
- Case 3: u is not **red** and the g to n path is straight

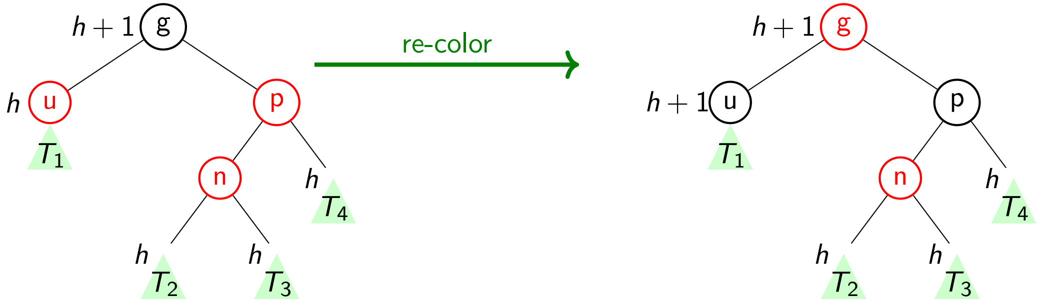
"not **red**" means either **black** node or null node.

No transformation should change the black height of g.

Exercise 7.5

Why must g exist and be black?

Case 1: The uncle is red



In the subtree of g, there is no change in the black height and no **red-red** violation.

Now g is red. We have three possibilities: the parent of g is **black**, the parent of g is **red**, and g is the root.

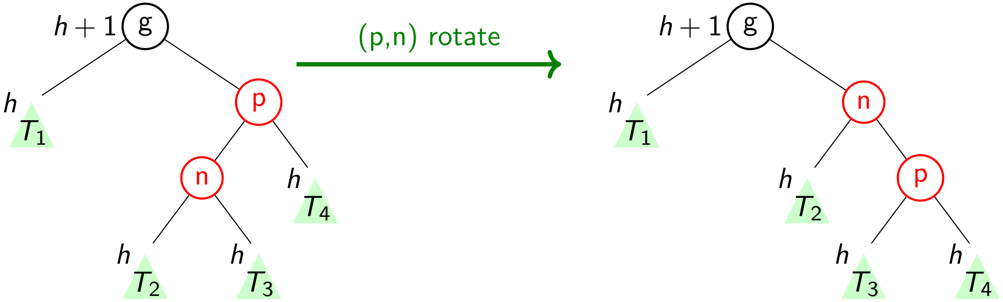
Exercise 7.6

What do we do in each of the possibilities?

Commentary: Possibility 1: Nothing. Possibility 2: We have a red-red violation a level up and need to apply the transformations there. Possibility 3: turn g back to black.

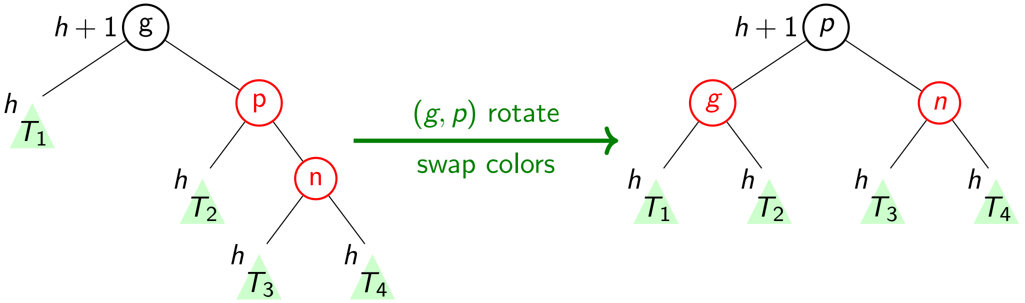
Case 2: The uncle is not red and the path to the grandparent is not straight

straight means $left^2(parent^2(n)) = n$ or $right^2(parent^2(n)) = n$



This transformation does not resolve the violation but converts the violation to case 3.

Case 3: The uncle is not red and the path to the grandparent is straight



The transformation removes the red-red violation.

Exercise 7.7

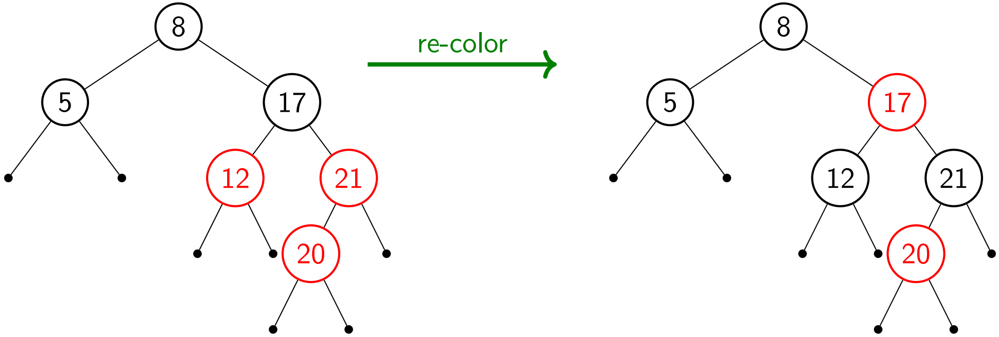
a. Why are the roots of T_2, T_3, and T_4 not **red**?

b. Show that if the root of T_1 is **red** then the above operation does not work.

Example: red-red correction case 1

Example 7.6

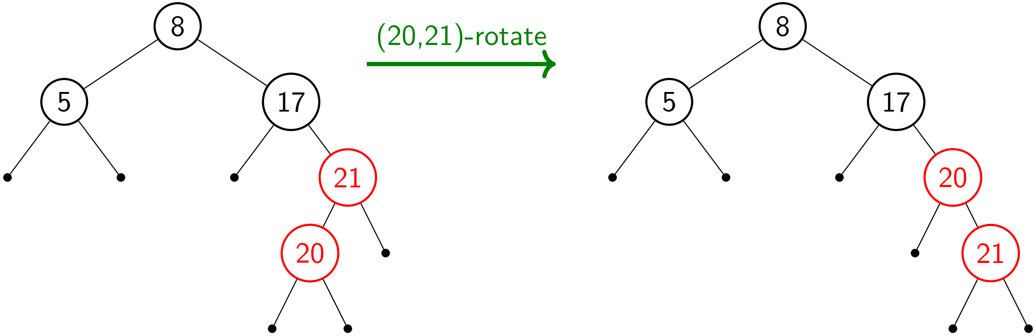
We just inserted 20 in the following tree. We need to apply case 1 to obtain a red-black tree.



Example: red-red correction case 2

Example 7.7

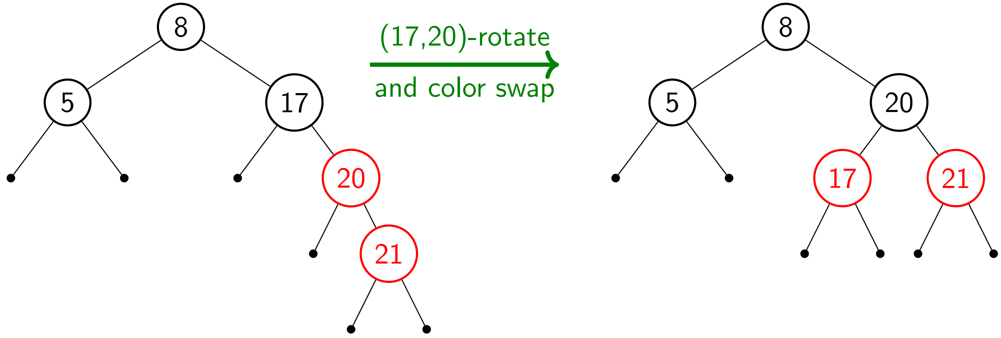
Consider the following example. We are attempting to insert 20. We apply case 2 to move towards a red-black tree.



The above is not a red-black tree. Furthermore, we need to apply case 3 to obtain the red-black tree.

Example: red-red correction case 3 (continued)

We apply case three as follows.



Summary of insertion

Insert like BST, make the new node **red**, and run the following algorithm.

Algorithm 7.1: RedRedRepair(Node n)

```
1 ASSUME(there is no red-red violation at any other node);
2 if $n$ is root then n.color := black; return;
3 p := parent(n);
4 if $p$ is black then return;
5 g := parent(p); u := sibling(p);
6 if u.color is red then  g.color:=red; u.color:=p.color:=black; RedRedRepair(g); return; // Case 1 ;
7 if  $left^2(g) \neq n$ and $right^2(g) \neq n$ then  Rotate(p,n);RedRedRepair(p);return;        //Case 2->3 ;
8 Rotate(g,p); g.color,p.color := p.color,g.color;                                           //Case 3  ;
```

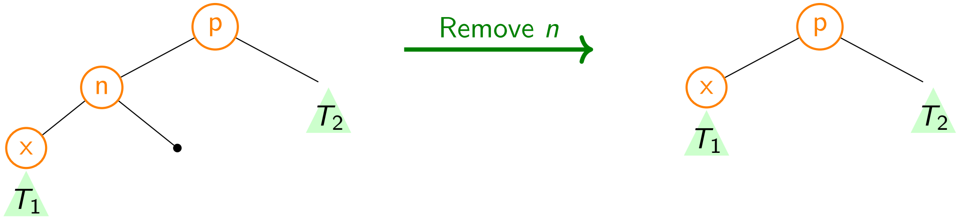
Commentary: What is the intuitive difference between cases 1 and 3? Answer: Case 1 reduces the total number of red nodes in a subtree. Case 3 balances if there are too many red nodes on one side of the subtree.

Topic 7.4

Deletion in red-black tree**

BST deletion in red-black tree

- Delete a node as if it is a binary search tree.
- Recall: In the BST deletion we always delete a node n with at most one non-null child.



x can be either a null or non-null node.

What can go wrong with a red-black tree?

Since a child x of n takes the role of n, we need to check if x can replace n.

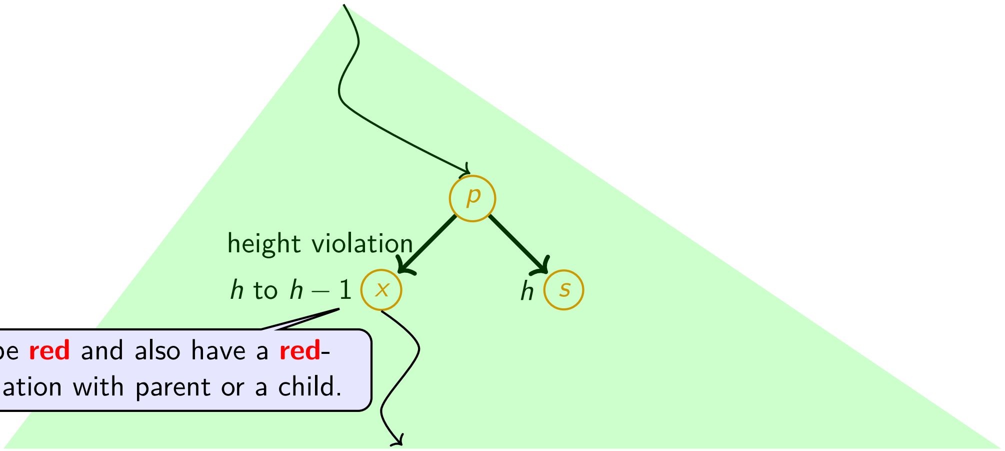
- If n was **red**, no violations occur. (Why?)
- If n was **black**, $\underbrace{bh(x) = bh(n) - 1}_{\text{black height violation}}$, or it is possible that $\underbrace{\text{both } x \text{ and } p \text{ are } \textbf{red}}_{\text{red-red violation}}$.

The leaves of the subtree rooted at x will have one less black depth.

Commentary: The or in the second bullet point is not xor. Both violations are possible at the same time.

Violation removal procedure

To remove the violation at x, we may recolor and rotate around x, which may push the violation to the parent. Therefore, we assume that the violation is somewhere in the middle of the tree.



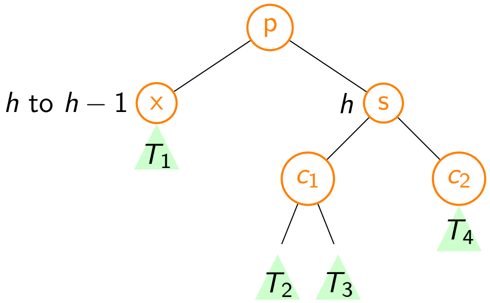
x may be **red** and also have a **red-red** violation with parent or a child.

Orange means that we need to consider all possible colors of the nodes.

Violation pattern

Commentary: Please check that if cases are exhaustive. You will notice that the case where x is not **red**, x is not root, and s is null-leaf is missing. The exercise in the slide proves that the case is not possible.

After the deletion, we may need to consider the following five nodes around x.



We correct the violation either by rotation or re-coloring.

There are six cases

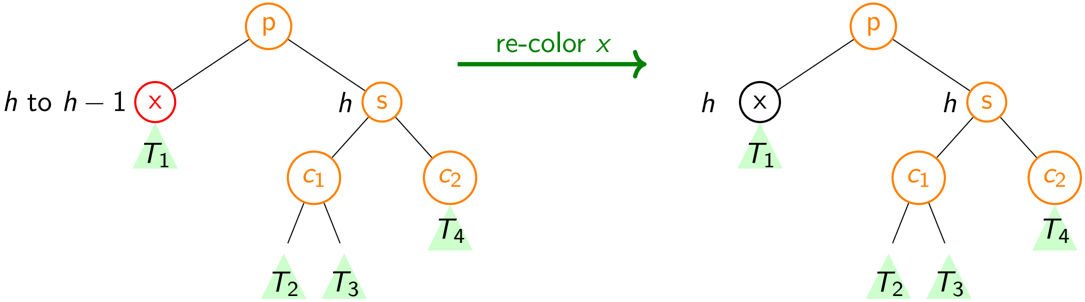
1. x is **red**
2. x is not **red** and root.
3. x is not **red** and s is **red**
4. x is not **red** and s is **black**
   - 4.1 c_2 is **red**
   - 4.2 c_2 is not **red** and c_1 is **red**
   - 4.3 c_2 is not **red** and c_1 is not **red**

The goal is to restore the black height of p.
Trick: find a **red** node and turn it **black**.

Exercise 7.8

Show if x is not root and not **red**, s is non-null.

Case 1: x is **red**

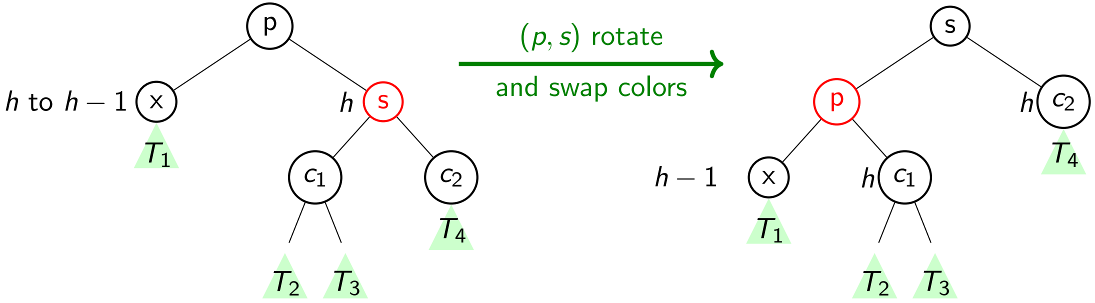


Violation solved!

Case 2: x is not **red** and root.

Do nothing.

Case 3: x is not **red** and the sibling of x is **red**

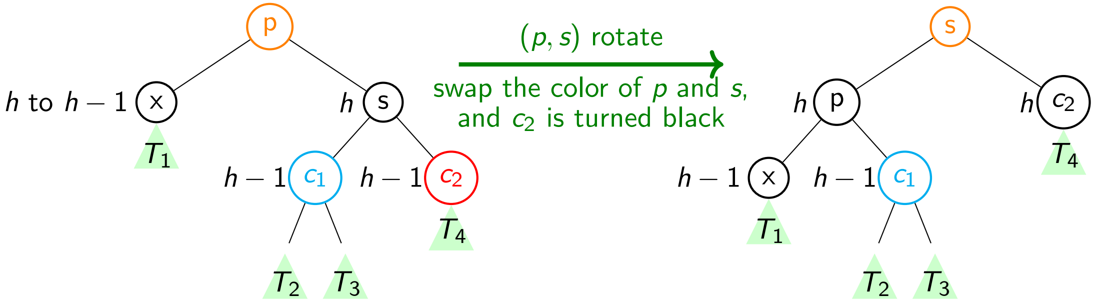


The transformation does not solve the height violation at the parent of x but changes the sibling of x from **red** to **black**.

Exercise 7.9
Why must p, c_1, and c_2 be non-null and **black**?

Case 4.1: x is not **red**, the sibling of x is **black**, and the far nephew is **red**

The color of p and c_1 does not matter.

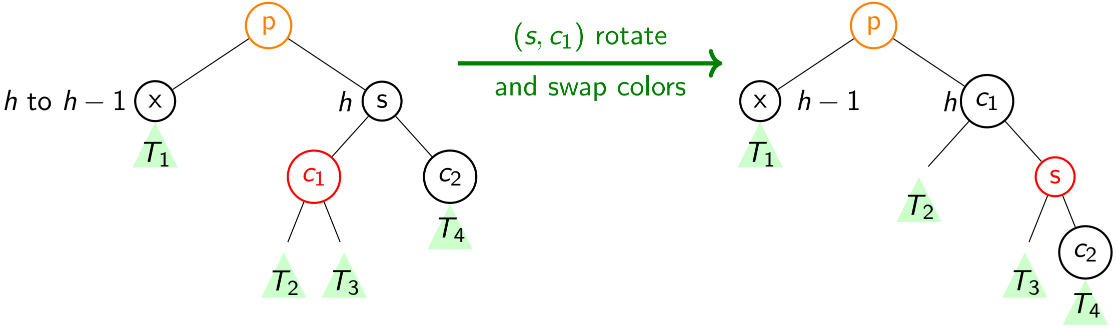


The above transformation solves the black height violation.

Exercise 7.10

Write the above case if x is the right child of p.

Case 4.2: x is not **red** and the sibling is **black**, and the near and far nephews are **red** and not **red** respectively



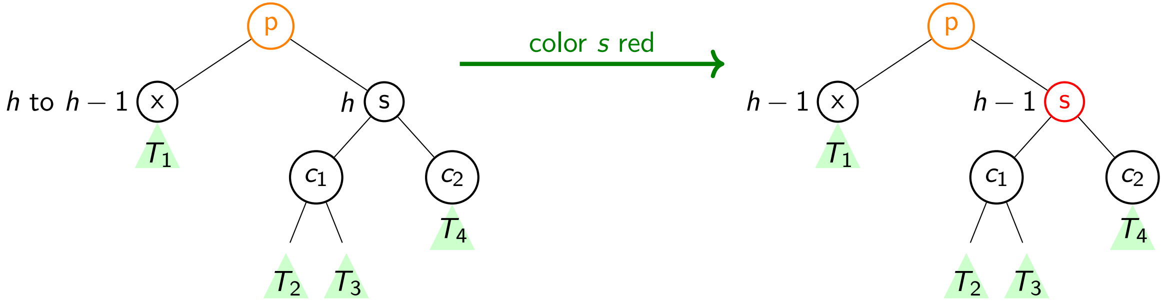
The above transformation does not solve the height violation. It changes the right child of the sibling from not **red** to **red**, which is the case 4.1.

Exercise 7.11

a. Why can case 4.1 transformation not be applied to case 4.2?

b. Write the above case if x is the right child of p.

Case 4.3: x is not **red**, the sibling of x is **black**, and the nephews of x are not **red**



The above transformation reduces $bh(p)$ by 1. There may be a potential violation at p, which is at the lower level. If p is red, then p and s will also have red-red violation.

We apply the case analysis again at node p. The only case that kicks the can upstairs!!
All cases are covered.

Structure among cases

- Cases 1, 2, and 4.1 solve the violation at the node.
- Case 3 turns the violation into 4.1, 4.2, or $(4.3 \rightarrow 1)$.
- Case 4.2 turns the violation into 4.1.
- Case 4.3 moves the violation from x to its parent p.

Summary of deletion

Delete *n* like BST, if there is a height violation at the parent of *x* then run the following algorithm.

Algorithm 7.2: HeightRepair(Node x)

```
ASSUME(there is no red-red violation at any other node);
ASSUME(there is no height violation below x);
if x is red then  x.color := black; return;                                          // Case 1;
if x is root then  return;                                                            // Case 2;
p := parent(x); s := sibling(x);
c1 := (x = left(p)) ? left(s) : right(s);                                             // near nephew;
c2 := (x = left(p)) ? right(s) : left(s);                                             // far nephew;
if s is red then  RotateAndSwapColor(p,s); HeightRepair(x); return;     // Case 3 → 4.1, 4.2, or (4.3→1);
if c2 is red then  RotateAndSwapColor(p,s); color(c2) := black; return;               // Case 4.1;
if c1 is red then  RotateAndSwapColor(s,c1);HeightRepair(x); return;                  // Case 4.2 → 4.1;
color(s) := red;
HeightRepair(p);                                                                      // Case 4.3
```

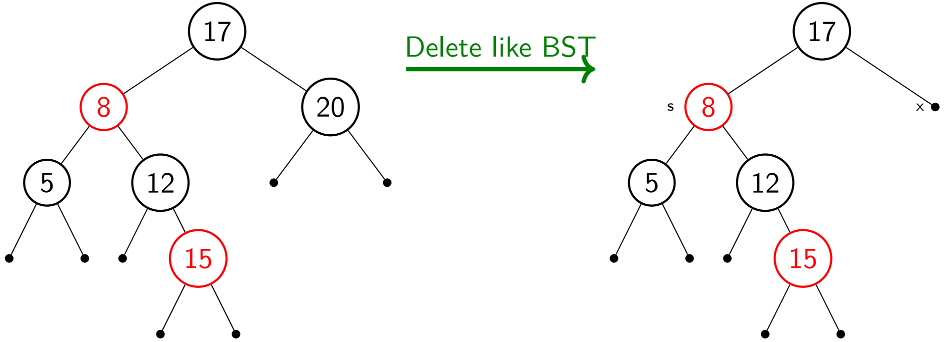
Exercise 7.12

Is the above a tail-recursive function?

Example: deletion in Red-black tree

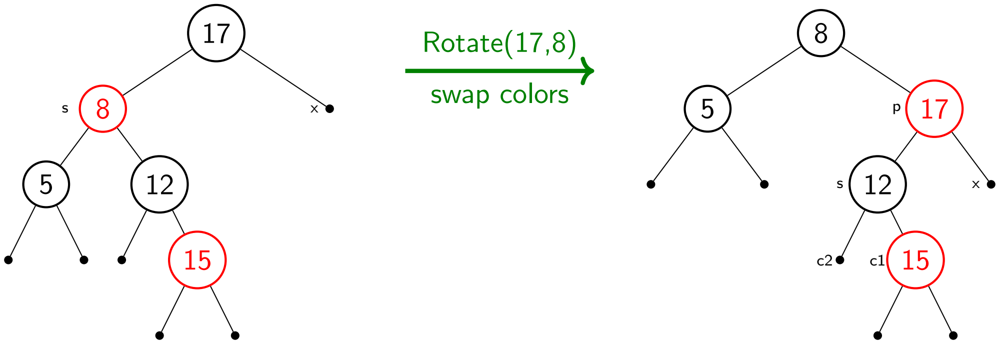
Example 7.8

Let us delete 20 in the following red black tree



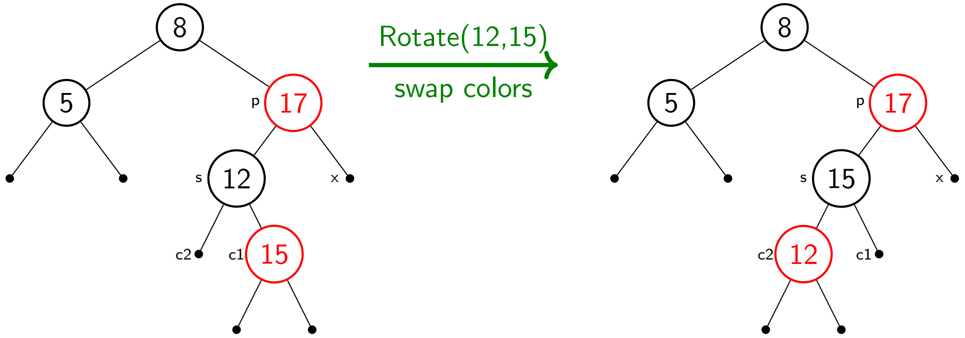
Node x is in Case 3, where the sibling s is **red**.

Example: deletion in red-black tree (continued) – applying Case 3



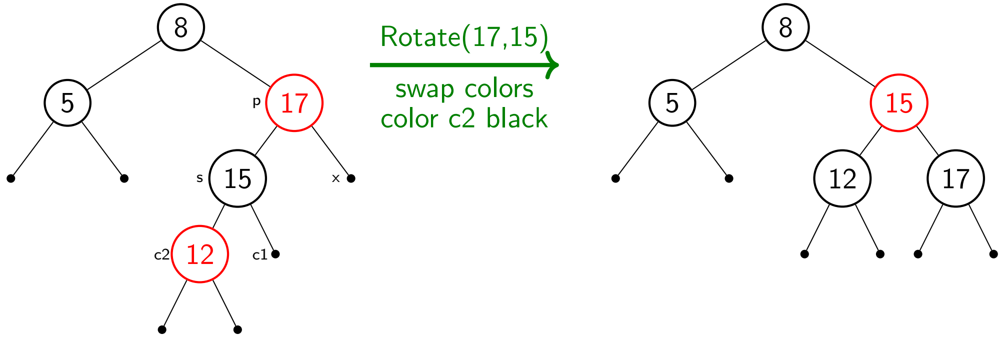
Now node x is in Case 4.2, where the sibling s is not **red**, nephew c2 is not **red**, nephew c1 is **red**.

Example: deletion in red-black tree (continued) – applying Case 4.2



Now node x is in Case 4.1, where the sibling s is not **red** and nephew c2 is **red**.

Example: deletion in red-black tree (continued) – applying Case 4.1

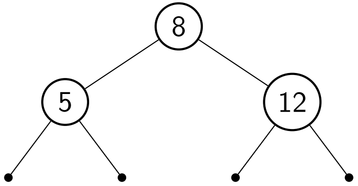


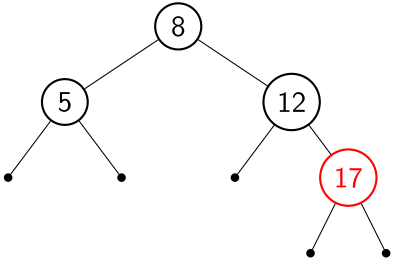
Height imbalance due to the deletion is repaired.

Exercise: identify the case for deletion and apply deletion

Exercise 7.13

On deletion of 12 in the following red-black trees, which case of deletion will be applied?





Summary of deletion

1. Delete like BST. There may be a black height violation at the child of the deleted node.
2. While case 4.3 occurs, re-color nodes and move up the black height violation.
3. For all the other cases, we rotate or re-color, and the violation is finished.

Topic 7.5

Tutorial problems

Exercise: validity of rotation

Exercise 7.14

Prove that after rotation the resulting tree is a binary search tree.

Exercise: sorted insert

Exercise 7.15

Insert sorted numbers 1,2,3,..., 10 into an empty red-black tree. Show all intermediate red-black trees.

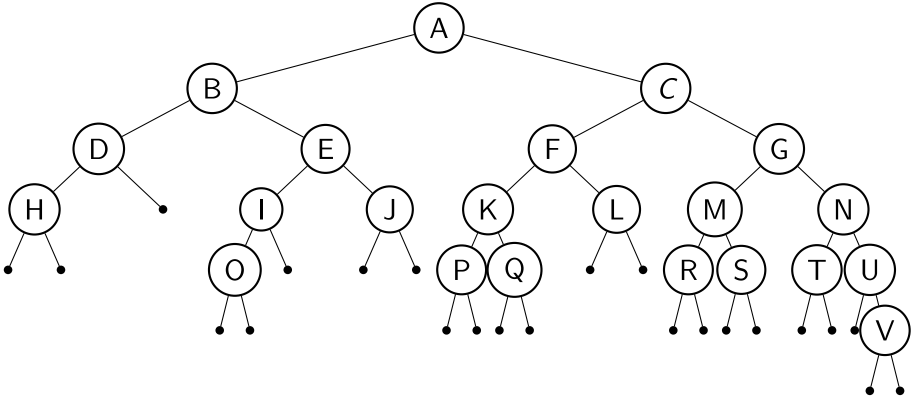
Exercise 7.16

Exact number of rotations needed to insert sorted numbers $1, \ldots, n$ into an empty red-black tree.**

Insert and delete

Exercise 7.17

Consider the tree below. Can it be colored and turned into a red-black tree? If we wish to store the set 1, . . . , 22, label each node with the correct number. Now add 23 to the set and then delete 1. Also, do the same in the reverse order. Are the answers the same? When will the answers be the same?



Exercise: Hash table vs red-black tree

Exercise 7.18

a. Give running time complexities of delete, insert, and search in the red-black tree.

b. What are the advantages of the red-black tree compared to the Hash table, where every operation (search, insert, delete) is almost constant time?

Topic 7.6

Problems

True or False

Exercise 7.19

Mark the following statements True / False and also provide justification.

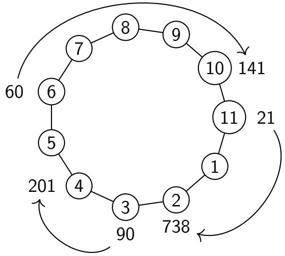
1. Each black node of a red-black tree must have a red child.
2. Unordered maps in C++ use red-black trees.
3. An insertion in red-black tree needs at most two rotations.

Exercise: consistent hashing** (midsem 2024)

Exercise 7.20

Let us suppose we want to store n keys on m servers. The servers have IDs. We may use a hash function h to map the keys to servers. Both keys and IDs of the servers are hashed. The computed hash values are placed on a *ring*, like a clock. Some of the points on the ring are the servers and some are keys. A key is stored on the closest server in the clockwise direction. Each time a server is added/removed the keys are moved according to the above rule.

Example: Let $h(x) = 1 + x\%11$. We want to store keys $21, 60$, and 90 on three server with IDs $141, 201$, and 738. The hash values of the keys are $11, 6, 3$ and the hash values of the servers are $10, 4, 2$. The keys $21, 60$, and 90 will be stored in servers 738, 141, and 201 respectively.



a. Give an efficient algorithm for adding/removing keys on the servers.

b. Give an efficient algorithm for implementing add/remove of servers.

Commentary: **Solution:** To implement the above scheme. We store the server hashes in a red black tree R.

```
findNext(k,s,s'){ if( s=Null) return s';if(key(s) < k) return findNext(k,key(s),s'); else return findNext(k,left(s),s);}
findServer(k){s = findNext(k,R.root,Null)  if(s == Null) s = minimum(R.root); if(s == Null ) Return Null; Return value(s)}
AddKey(k){ s = findServer(h(k));storeInServer(k,s)} RemoveKey(k){ s = findServer(h(k),R.root); removeInServer(k,s)}
AddServer(s){ s' := h(s) R.Insert( (s',s) ); t = nextServer(s'); for k in GreaterKeysOnServer(s',t) {removeInServer(k,t);storeInServer(k,s)} }
RemoveServer(s){ s' := h(s) R.Delete(s'); t = nextServer(s'); for k in KeysOnServer(s) { removeInserver(k,s);storeInServer(k,t); } }
```

Assume red-black Insert takes (key,value) pair as input and no two servers have collision, which can be easily avoided by renaming the IDs. Keys are again stored as Red-black tree on the servers such that we can efficiently implement GreaterKeysOnServer. removeInServer are storeInServer are trivial calls to insert/delete of RB Tree.

Excercise: Is BST red-black colorable?**

Exercise 7.21

Given a BST, can we check if there exists a colouring which makes it a valid RB-Tree?

Commentary: **Solution:** https://cs.stackexchange.com/questions/10990/colour-a-binary-tree-to-be-a-red-black-tree

Topic 7.7

Extra slides: AVL trees (In GATE/GRE syllabus)

AVL (Adelson, Velsky, and Landis) tree

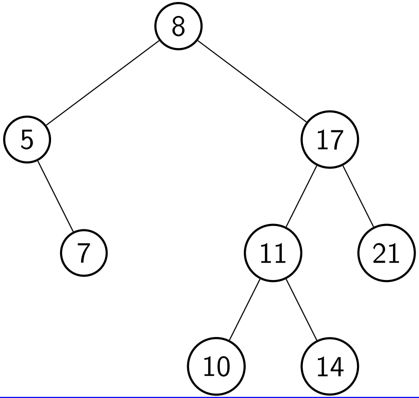
Definition 7.3

An AVL tree is a binary search tree such that for each node *n*

$$|height(right(n)) - height(left(n))| \leq 1.$$

Example 7.9

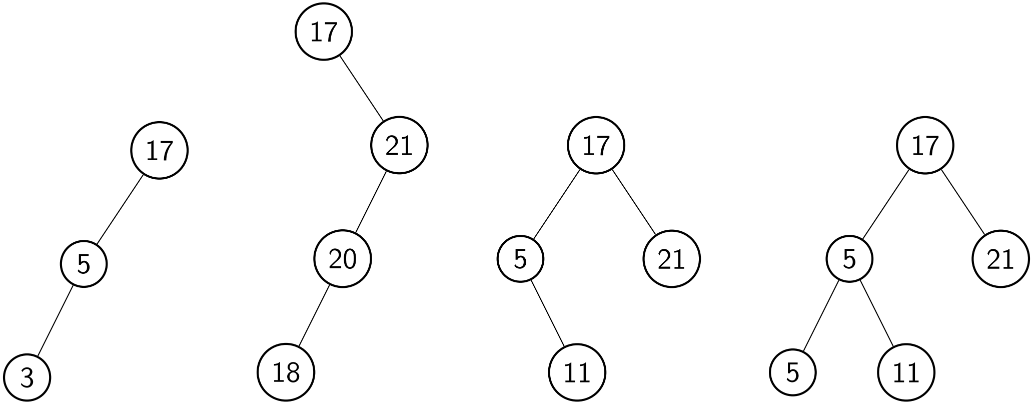
An example of an AVL tree.



Exercise: Identify the AVL trees

Exercise 7.22

Which of the following are AVL trees?



Topic 7.8

Height of AVL tree

AVL tree height

Theorem 7.1

The height of an AVL tree T having n nodes is $O(\log n)$.

Proof.

Let $n(h)$ be the minimum number of nodes for height h.

Base case:

$n(1) = 1$ and $n(2) = 2$.

Induction step:

Consider an AVL tree with height $h \geq 3$. In the minimum case, one child will have a height of $h-1$ and the other child will have a height of $h-2$. (Why?)

Therefore, $n(h) = 1 + n(h-1) + n(h-2)$. ...

Commentary: We need to show that $n(h) > n(h-1)$ is monotonous. Ideally, $n(h) = 1 + n(h-1) + min(n(h-2), n(h-1))$. This proves that $n(h) > n(h-1)$.

AVL tree height(2)

Proof(continued.)

Since $n(h-1) > n(h-2)$,

$$n(h) > 2n(h-2).$$

Therefore,

$$n(h) > 2^i n(h-2i).$$

For $i = h/2 - 1$(Why?),

$$n(h) > 2^{h/2-1} n(2) = 2^{h/2}.$$

Therefore,

$$h < 2 \log n(h).$$

Therefore, the height of an AVL tree is $O(\log n)$. (Why?)

Commentary: Here is the explanation of the last step. Consider an AVL tree with m nodes and h height. By definition, $h(n) \leq m$. Since $h < 2\log n(h)$, $h < 2\log m$. Therefore, h is $O(\log m)$.

Closest leaf

Theorem 7.2

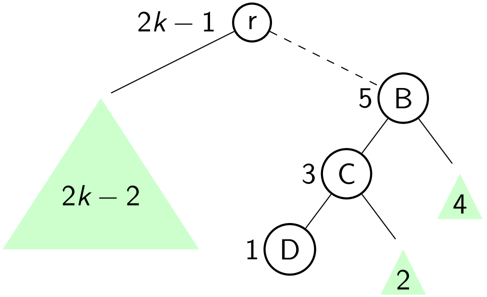
Let T be an AVL tree. Let the level of the closest leaf to the root of T is k.

$$height(T) \leq 2k - 1$$

Proof.

Let D be the closest leaf of the tree.

- The height of $right(C)$ cannot be more than 2. (Why?)
- Therefore, the maximum height of C is 3.
- Therefore, the maximum height of $right(B)$ is 4.
- Therefore, the maximum height of B is 5.
- Continuing the argument, the maximum height of root r is $2k - 1$.



□

A part of AVL is a complete tree

Theorem 7.3

Let T be an AVL tree. Let the level of the closest leaf to the root of T is k. Upto level $k-2$ all nodes have two children.

Proof.

A node at level $k-2-i$ cannot be a leaf for $i \geq 0$. (Why?)

Let us assume that a node n at level $k-2-i$ has a single child n'.

The height of n' cannot be more than 1. (Why?)

Therefore, n' is a leaf. Contradiction. $\square$

Exercise 7.23

Show T has at least 2^{k-1} nodes.

Another proof of tree height bound

Let T have n nodes and the height of T be h.

We know the following from the previous theorems.

- $n \geq 2^{k-1}$, and
- $2k - 1 \geq h$.

Therefore,

$$n \geq 2^{k-1} \geq 2^{(h-1)/2}$$

Exercise 7.24

What is the maximum number of nodes given height h?

Problem: A sharper bound for the AVL tree

Exercise 7.25

a. Find largest c such that $c^{k-2} + c^{k-1} \geq c^k$

b. Recall $n(h) = 1 + n(h-1) + n(h-2)$. Let c_0 be the largest c. Show that $n(h) \geq c^{h-1}$.

c. Prove that the above bound is a sharper bound than our earlier proof.

Topic 7.9

Insertion and deletion in AVL trees

Insert and delete

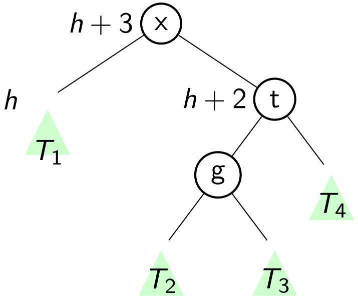
Insert and delete like BST.

At most a path to one node may have height imbalances of 2. (Why?)

We have to repair height imbalances by rotations around the deepest imbalanced node.

Rebalancing AVL trees

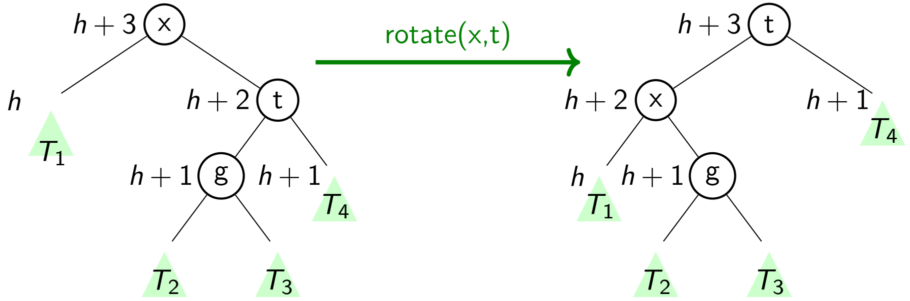
Let x be the deepest imbalanced node. Let t be the taller child. Let g be the grandchild via t that is not on the straight path from x.



Three cases:

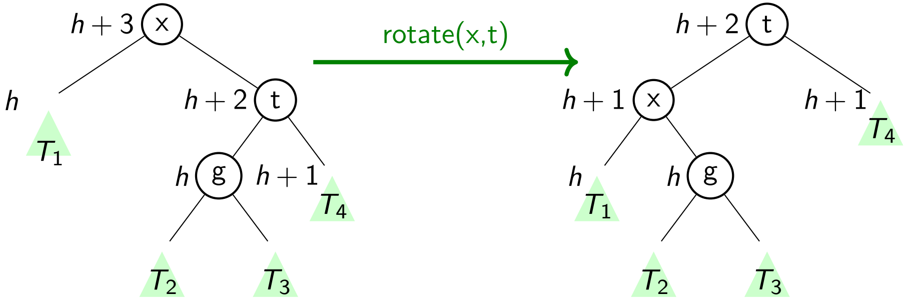
1. Case 1: Height of g is $h+1$ and T_4 is $h+1$.
2. Case 2: Height of g is h and T_4 is $h+1$.
3. Case 3: Height of g is $h+1$ and T_4 is h.

Case 1: Both grandchildren via t have height $h+1$



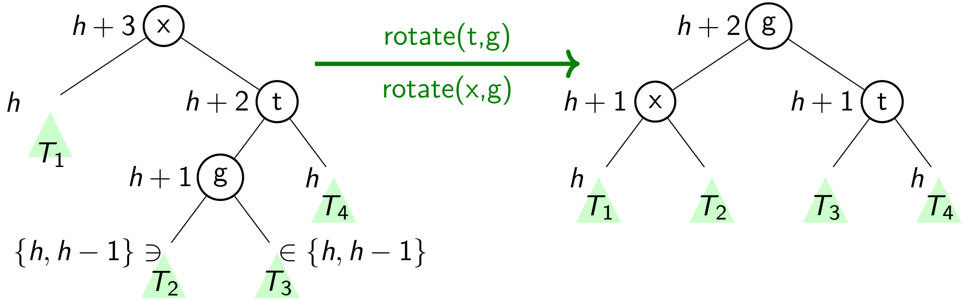
The imbalance in the subtree is repaired. We check the parent of t.

Case 2: Right-left grandchild has height h



Imbalance is repaired. But, the parent of t may need repair.

Case 3: Right right grandchild has height h



Imbalance is repaired. But, the parent may need repair.

Complexity of insertion/deletion

Exercise 7.26

a. What is the bound on the number of rotations for a single insert/delete?

b. Compare the bounds with RB trees insertion/deletion.

c. Which definition is more strict RB or AVL? Or, are they incomparable?

End of Lecture 7