CS 105: DIC on Discrete Structures

Instructor : S. Akshay

Aug 08, 2023

Lecture 02 – Types of proofs, Mathematical Induction

Logistics and recap

Course material, references are being posted at

- `http://www.cse.iitb.ac.in/~akshayss/teaching.html`
- Piazza has been set up and you must have got the invites. Please join asap.

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Recap of last lecture

- What are discrete structures, course outline.
- Chapter 1: proofs and structures. Propositions, theorems.
- Theorems and proofs.

Theorems and proofs

A theorem is a proposition which can be shown true

Prove the following theorems.

1. For all $a, b, c \in \mathbb{R}^{\geq 0}$, if $a^2 + b^2 = c^2$, then $a + b \geq c$
2. If 6 is prime, then $6^2 = 30$.
3. For all $x \in \mathbb{Z}$, x is even iff $x + x^2 - x^3$ is even.

4. There are infinitely many prime numbers.
5. There exist irrational numbers x, y such that x^y is rational.
6. For all $n \in \mathbb{N}$, $n! \leq n^n$.
7. There does not exist a (input-free) C-program which will always determine whether an arbitrary (input-free) C-program will halt.

Theorems and proofs

Contrapositive and converse

- The contrapositive of "if A then B" is "if $\neg B$ then $\neg A$".
- A statement is logically equivalent to its contrapositive, i.e., it suffices to show one to imply the other.
- To show A iff B, you have to show A implies B and conversely, B implies A.
- Note the difference between contrapositive and converse.

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(H.W): Post a constructive proof of this theorem on piazza.

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6. For all $n \in \mathbb{N}$, $n! \leq n^n$. – next!
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- Rules of inference: Logic, e.g.,
  - if p is true, and p implies q, then q is true.
  - if p is true, then $p \vee q$ is true.
  - if p is true and q is true, then $p \wedge q$ is true.
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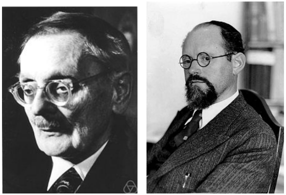
- Axioms: Peano's axioms, Euclid's axioms.

Axioms

(a) Euclid

(b) G. Peano



(c) Zermelo-Fraenkel

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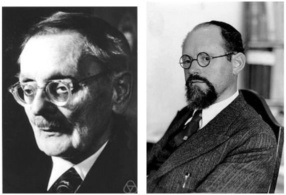

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(c) Zermelo-Fraenkel and Choice axioms (ZFC) are a small set of axioms from which most of mathematics can be inferred.

  - But proving even $2+2=4$ requires $>$ 20000 lines of proof!

- In this course, we will assume axioms, mostly from high school math (distributivity of numbers etc.).

Introducing the world of Mathematical Induction

Induction (Axiom)

Let $P(n)$ be a property of non-negative integers. If

- $P(0)$ is true (Base case)
- for all $k \geq 0$, $P(k) \implies P(k+1)$ (Induction Step)

then $P(n)$ is true for all $n \in \mathbb{N}$.

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Proof by induction: we will show for all $n \geq 2$, $n! < n^n$

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   $(k+1)! = k! \cdot (k+1) \leq k^k(k+1)$ (by Induction Hypothesis)

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$$\begin{aligned}(k+1)! = k! \cdot (k+1) &\leq k^k (k+1) \text{ (by Induction Hypothesis)} \\ &< (k+1)^k \cdot (k+1) = (k+1)^{(k+1)}\end{aligned}$$

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$$(k+1)! = k! \cdot (k+1) \leq k^k (k+1) \text{ (by Induction Hypothesis)}$$
$$< (k+1)^k \cdot (k+1) = (k+1)^{(k+1)}$$

4. Hence by induction, we conclude that for all $n \geq 2$, $n! < n^n$.

Examples by induction (H.W)

1. Summations:
   1.1 $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$.
   1.2 $1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1} n^2 = (-1)^{n-1} \frac{n(n+1)}{2}$

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– "Proof technique" rather than a "Solution technique" as it requires a good guess of the answer.

Interesting fallacy in using induction!

Conjecture: All horses have the same colour.

"Proof" by induction on number of horses:

1. Base Case ($n = 1$) The case with one horse is trivial.
2. Induction Hypothesis Assume for $n = k \geq 1$, i.e., any set of $k (\geq 1)$ horses has same color.
3. Induction Step We want to show any set of $k + 1$ horses have same color. Consider such a set, say $1, \ldots, k + 1$.
   (A) First, consider horses $1, \ldots, k$. By induction hypothesis, they have same color.
   (B) Next, consider horses $2, \ldots, k + 1$. By induction hypothesis, they have same color.
   (C) Therefore, 1 has same color as 2 (by A) and 2 has same color as $k + 1$ (by B), implies all $k + 1$ have same color.
4. Thus, by induction, we conclude that for all $n \geq 1$, any set of n horses has the same color. □

Where is the bug?