CS 105: Department Introductory Course on Discrete Structures

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Lecture 05 – Basic Mathematical Structures

Recap of last three lectures

Chapter 1: Mathematical reasoning

- Propositions, predicates.
- Axioms, Theorems and Types of proofs: contradiction, contrapositive, etc.
- Principle of Mathematical Induction
- Well-ordering principle.

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- Formally writing a proof.
- Proof by Well-Ordering Principle.

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   lhs$= 1+2+\ldots+k+(k+1) = (1+2+\ldots k)+(k+1)$
   $= \frac{k(k+1)}{2} + (k+1)$ (By Induction Hypothesis)
   $= \frac{k(k+1)+2(k+1)}{2} = \frac{(k+2)(k+1)}{2} = rhs.$

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4. Hence by induction we can conclude for all $n \in \mathbb{N}, n \geq 1$.

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9. Implies $k' \notin S$. A contradiction.

From proofs to structures

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Next: Chapter 2: Basic Mathematical Structures

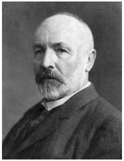
- Finite and infinite sets, Functions
- Relations

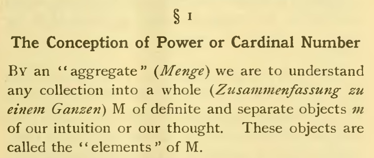
Sets

What is a set?

- A set is an unordered collection of objects.
- The objects in a set are called its elements.

Sets





§ I

The Conception of Power or Cardinal Number

By an "aggregate" (*Menge*) we are to understand any collection into a whole (*Zusammenfassung zu einem Ganzen*) M of definite and separate objects *m* of our intuition or our thought. These objects are called the "elements" of M.

Figure: Georg Cantor (1845-1918); extract

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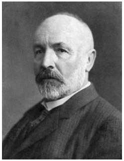

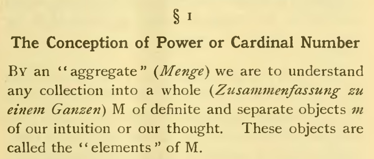

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More formally,

Let P be a property. Any collection of objects that are defined by (or satisfy) P is a set, i.e., $S = \{x \mid P(x)\}$.

Some simple boring stuff about sets

Examples and properties

- We have already seen examples: $\mathbb{Z}, \mathbb{N}, \mathbb{R}$, set of all horses,...
- Let A, B be two sets. Recall the usual definitions:
  - Equality $A = B$, Subset $A \subseteq B$,
  - Cartesian product $A \times B = \{(a, b) \mid a \in A, b \in B\}$
  - Union $A \cup B = \{x \mid a \in A \text{ or } b \in B\}$
  - Intersection $A \cap B = \{x \mid a \in A \text{ and } b \in B\}$
  - Empty set ϕ,
  - Power set of $A = \mathcal{P}(A)$= set of all subsets of A.
  - If U is the universe, then the complement of A, $\bar{A} = A^c = \{x \in U \mid x \notin A\}$.

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So, what is the difference between $\{\emptyset\}$ and $\emptyset$?

Not so simple...

A barber is a man in town who only shaves those who don't shave themselves.

Barber's paradox: Does the barber shave himself?

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Then if $S \in S$, then $S \notin S$ and if $S \notin S$, then $S \in S$!

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How do you resolve this?

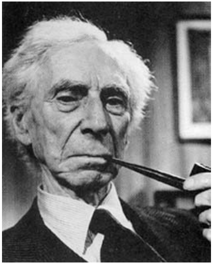

Figure: Bertrand Russell (1872-1970)

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Start with a few objects defined. Then for a set A and a property P, $S = \{x \in A \mid P(x)\}$ is a set.

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Why does this definition get rid of Russell's paradox?

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- if $(S \in S)$: from the definition of S, $S \in A$ and $S \notin S$, which is a contradiction.
- if $(S \notin S)$: from the definition, either $S \notin A$ or $S \in S$. But we have assumed that $S \notin S$. Hence, $S \notin A$. No contradiction!