CS 105: Department Introductory Course on Discrete Structures

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Aug 21, 2023

Lecture 06 – Basic Mathematical Structures

Sets and functions

A Quick Recap

Five lectures completed

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- Week 1
  1. Propositions, Predicates, Theorems.
  2. Types of proofs; contradiction and contrapositive; axioms.
  3. Induction and the Well-Ordering Principle.
- Week 2
  4. Strong Induction and its applications.
  5. Basic mathematical structures: numbers and sets

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  5. Basic mathematical structures: numbers and sets

Two problem-sheets released

1. 9 questions on Basic proofs, induction, WOP
2. 4 questions on More basic proofs and Strong Induction.

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- For two finite sets, this is easy, just count the number of elements and compare them!
- But what about two infinite sets?
- Example: {set of all even natural numbers} vs $\mathbb{N}$ vs $\mathbb{Q}$ vs $\mathbb{R}$
- Turns out we need functions... but first...

Hilbert's hotel

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2. What if infinitely many more guests arrive?
3. What if infinitely many more trains with infinitely many more guests arrive? (H.W)

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Let A, B be two sets. A function f from A to B is an assignment of exactly one element of B to each element of A.

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Formally, $f : A \to B$ is a subset R of pairs $A \times B$ such that

(i) $\forall a \in A, \exists b \in B$ such that $(a, b) \in R$, and

(ii) if $(a, b) \in R$ and $(a, c) \in R$, then $b = c$.

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- We write $f(a) = b$ and call b the image of a.
- $\mathit{Range}(f) = \{b \in B \mid \exists a \in A \text{ s.t. } f(a) = b\}$, $\mathit{Domain}(f) = A$

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Composition of functions

- If $g : A \to B$ and $f : B \to C$, then $f \circ g : A \to C$ is defined by $f \circ g(x) = f(g(x))$.

- Defined only if $Range(g) \subseteq Domain(f)$.
- Qn: if $f(x) = x^2$, $g(x) = x - x^3$ with $f, g : \mathbb{R} \to \mathbb{R}$, what is $f \circ g(x), g \circ f(x)$?

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Composition of functions is associative

- If $h : A \to B$ and $g : B \to C$ and $f : C \to D$, then $f \circ (g \circ h) = (f \circ g) \circ h$.

Check it! (H.W.)

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Inverse of a function

- If $f : A \to B$ is a function, then its inverse is the function $f^{-1} : B \to A$ defined by $f^{-1}(b) = a$ if $f(a) = b$.

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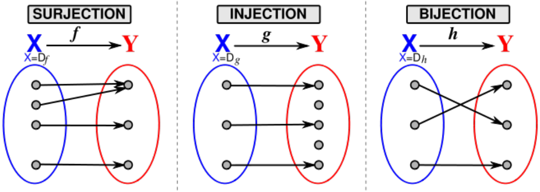
Inverse of a function

- If $f : A \to B$ is a function, then its inverse is the function $f^{-1} : B \to A$ defined by $f^{-1}(b) = a$ if $f(a) = b$. Does the inverse always exist?

Comparing (finite and infinite) sets



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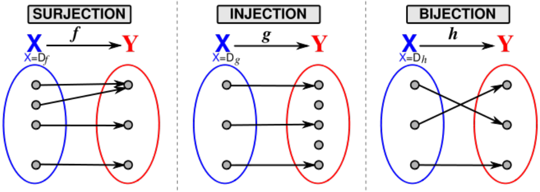

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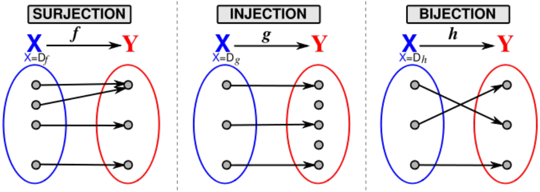

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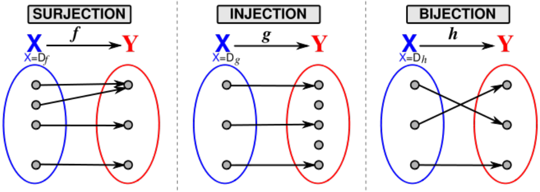

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If f is a bijection, then its inverse function exists and $f \circ f^{-1} = f^{-1} \circ f = \text{id}$

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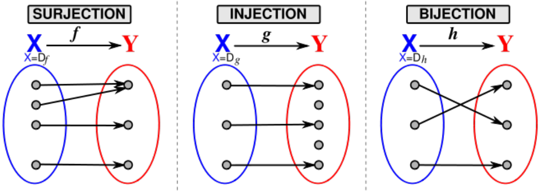

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Qns

1. $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x) = x^2$.
2. $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ such that $f(x) = x^2$.

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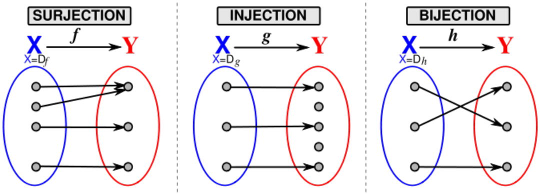

- Surjective or onto: $f : A \to B$ is surjective if $\forall y \in B$, $\exists x \in A$ such that $f(x) = y$. – If A, B finite, $|A| \geq |B|$
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Qns

1. $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x) = x^2$.
2. $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ such that $f(x) = x^2$.

Properties of finite and infinite sets

Relative notion of "size" using bijections

Thus, two finite/infinite sets have the same "size" iff there is a bijection between them.

- For finite sets, this is a property that can be shown.
- For infinite sets, it is a definition!

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Relative notion of "size" using bijections

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Similarities between finite and infinite sets

- $\exists$ **bij** from A to B and B to C, implies $\exists$ **bij** from A to C.
- $\exists$ **bij** from A to B, then $\exists$ **bij** from B to A.
- $\exists$ **surj** from A to B and $\exists$ **surj** B to A, implies $\exists$ **bij** from A to B.

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Differences between finite and infinite sets

- For finite sets, if A is a set and $b \notin A$, then $|A| \neq |A \cup \{b\}|$.
- What about infinite sets?

Difference between finite and infinite sets

Theorem

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Proof: essentially Hilbert's hotel but be careful...