CS 105: Department Introductory Course on Discrete Structures

Instructor : S. Akshay

Aug 24, 2023

Lecture 08 – Basic Mathematical Structures

Countable and Uncountable Sets

Countable and countably infinite sets

Definition

- Set C is called countably infinite, if there is a bijection from set C to $\mathbb{N}$.
- A set is countable if it is finite or countably infinite.

Examples: even numbers, number of horses,...

By previous corollary ($\exists$ surj from any infinite set to $\mathbb{N}$)

Countably infinite sets are the "smallest" infinite sets.

Some questions...

Are the following sets countable?

That is, is there a bijection from these sets to $\mathbb{N}$?

1. the set of all integers $\mathbb{Z}$
2. $\mathbb{N} \times \mathbb{N}$
3. $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$
4. the set of rationals $\mathbb{Q}$
5. the set of all (finite and infinite) subsets of $\mathbb{N}$
6. the set of all real numbers $\mathbb{R}$

Some questions...

Are the following sets countable?

That is, is there a bijection from these sets to $\mathbb{N}$?

1. the set of all integers $\mathbb{Z}$
2. $\mathbb{N} \times \mathbb{N}$
3. $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$
4. the set of rationals $\mathbb{Q}$
5. the set of all (finite and infinite) subsets of $\mathbb{N}$
6. the set of all real numbers $\mathbb{R}$

To show these it suffices to show that

- there is an injection from these sets to $\mathbb{N}$

Some questions...

Are the following sets countable?

That is, is there a bijection from these sets to $\mathbb{N}$?

1. the set of all integers $\mathbb{Z}$
2. $\mathbb{N} \times \mathbb{N}$
3. $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$
4. the set of rationals $\mathbb{Q}$
5. the set of all (finite and infinite) subsets of $\mathbb{N}$
6. the set of all real numbers $\mathbb{R}$

To show these it suffices to show that

- there is an injection from these sets to $\mathbb{N}$
- or there is a surjection from $\mathbb{N}$ (or any countable set) to these sets.

Unions of countable sets is countable

Let $A = \{a_0, \ldots, \}$ be a countably infinite set and B be a set. Then, is $A \cup B$ countable, under the following conditions?

1. $B = \{b_0\}$ is a singleton
2. $B = \{b_0, \ldots, b_n\}$ is a finite set
3. $B = \{b_0, \ldots\}$ is a countably infinite set

Unions of countable sets is countable

Let $A = \{a_0, \ldots,\}$ be a countably infinite set and B be a set. Then, is $A \cup B$ countable, under the following conditions?

1. $B = \{b_0\}$ is a singleton
2. $B = \{b_0, \ldots, b_n\}$ is a finite set
3. $B = \{b_0, \ldots\}$ is a countably infinite set
   Can we say $\{a_0, \ldots, b_0, \ldots\}$ is a countably infinite set?

Unions of countable sets is countable

Let $A = \{a_0, \ldots,\}$ be a countably infinite set and B be a set. Then, is $A \cup B$ countable, under the following conditions?

1. $B = \{b_0\}$ is a singleton
2. $B = \{b_0, \ldots, b_n\}$ is a finite set
3. $B = \{b_0, \ldots\}$ is a countably infinite set
   Can we say $\{a_0, \ldots, b_0, \ldots\}$ is a countably infinite set?

- But then what is the position of b_i (i.e., natural number corresponding to it)?

Unions of countable sets is countable

Let $A = \{a_0, \ldots,\}$ be a countably infinite set and B be a set. Then, is $A \cup B$ countable, under the following conditions?

1. $B = \{b_0\}$ is a singleton
2. $B = \{b_0, \ldots, b_n\}$ is a finite set
3. $B = \{b_0, \ldots\}$ is a countably infinite set
   Can we say $\{a_0, \ldots, b_0, \ldots\}$ is a countably infinite set?

- But then what is the position of b_i (i.e., natural number corresponding to it)?
- Rather, choose $\{a_0, b_0, a_1, b_1, \ldots\}$, then b_i is at $(2i + 1)^{th}$ position.

Unions of countable sets is countable

Let $A = \{a_0, \ldots,\}$ be a countably infinite set and B be a set. Then, is $A \cup B$ countable, under the following conditions?

1. $B = \{b_0\}$ is a singleton
2. $B = \{b_0, \ldots, b_n\}$ is a finite set
3. $B = \{b_0, \ldots\}$ is a countably infinite set
   Can we say $\{a_0, \ldots, b_0, \ldots\}$ is a countably infinite set?

- But then what is the position of b_i (i.e., natural number corresponding to it)?
- Rather, choose $\{a_0, b_0, a_1, b_1, \ldots\}$, then b_i is at $(2i+1)^{th}$ position.
- Formally, define a bijection $f : (A \cup B) \to \mathbb{N}$ by $f(a_i) = 2i$ and $f(b_i) = 2i + 1$

Unions of countable sets is countable

Let $A = \{a_0, \ldots,\}$ be a countably infinite set and B be a set. Then, is $A \cup B$ countable, under the following conditions?

1. $B = \{b_0\}$ is a singleton
2. $B = \{b_0, \ldots, b_n\}$ is a finite set
3. $B = \{b_0, \ldots\}$ is a countably infinite set
   Can we say $\{a_0, \ldots, b_0, \ldots\}$ is a countably infinite set?

- But then what is the position of b_i (i.e., natural number corresponding to it)?
- Rather, choose $\{a_0, b_0, a_1, b_1, \ldots\}$, then b_i is at $(2i+1)^{th}$ position.
- Formally, define a bijection $f : (A \cup B) \to \mathbb{N}$ by $f(a_i) = 2i$ and $f(b_i) = 2i + 1$
- Is this correct?

Products of countable sets are countable

Theorem: The cartesian product of two countably infinite sets is countably infinite

Products of countable sets are countable

Theorem: The cartesian product of two countably infinite sets is countably infinite

Proof: Let A, B be countably infinite. Find a way to "number" the elements in $A \times B = \{(a, b) \mid a \in A, b \in B\}$.

Products of countable sets are countable

Theorem: The cartesian product of two countably infinite sets is countably infinite

Proof: Let A, B be countably infinite. Find a way to "number" the elements in $A \times B = \{(a, b) \mid a \in A, b \in B\}$.

- That is, define a bijection from $A \times B$ to $\mathbb{N}$.

Products of countable sets are countable

Theorem: The cartesian product of two countably infinite sets is countably infinite

Proof: Let A, B be countably infinite. Find a way to "number" the elements in $A \times B = \{(a, b) \mid a \in A, b \in B\}$.

- That is, define a bijection from $A \times B$ to $\mathbb{N}$.

$$f(a_i, b_j) = \Big(\sum_{k=1}^{i+j} k\Big) + j + 1$$

Products of countable sets are countable

Theorem: The cartesian product of two countably infinite sets is countably infinite

Proof: Let A, B be countably infinite. Find a way to "number" the elements in $A \times B = \{(a, b) \mid a \in A, b \in B\}$.

- That is, define a bijection from $A \times B$ to $\mathbb{N}$.

$$f(a_i, b_j) = \Big(\sum_{k=1}^{i+j} k\Big) + j + 1$$

Corollaries

- $\mathbb{N} \times \mathbb{N}$, $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, $\mathbb{N} \times \mathbb{Z} \times \mathbb{N}$ are countable.

Products of countable sets are countable

Theorem: The cartesian product of two countably infinite sets is countably infinite

Proof: Let A, B be countably infinite. Find a way to "number" the elements in $A \times B = \{(a, b) \mid a \in A, b \in B\}$.

- That is, define a bijection from $A \times B$ to $\mathbb{N}$.

$$f(a_i, b_j) = \Big(\sum_{k=1}^{i+j} k\Big) + j + 1$$

Corollaries

- $\mathbb{N} \times \mathbb{N}$, $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, $\mathbb{N} \times \mathbb{Z} \times \mathbb{N}$ are countable.
- The set of (positive) rationals is countable.

Products of countable sets are countable

Theorem: The cartesian product of two countably infinite sets is countably infinite

Proof: Let A, B be countably infinite. Find a way to "number" the elements in $A \times B = \{(a, b) \mid a \in A, b \in B\}$.

- That is, define a bijection from $A \times B$ to $\mathbb{N}$.

$$f(a_i, b_j) = \Big(\sum_{k=1}^{i+j} k\Big) + j + 1$$

Corollaries

- $\mathbb{N} \times \mathbb{N}$, $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, $\mathbb{N} \times \mathbb{Z} \times \mathbb{N}$ are countable.
- The set of (positive) rationals is countable.

Hint: Show that $f(a, b) = \begin{cases} a/b \text{ if } b \neq 0 \\ 0 \text{ if } b = 0 \end{cases}$, is a surjection. How does the result follow?

Countable sets and functions

Are the following sets countable?

- the set of all integers $\mathbb{Z}$
- $\mathbb{N} \times \mathbb{N}$
- $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$
- the set of rationals $\mathbb{Q}$
- the set of all (finite and infinite) subsets of $\mathbb{N}$
- the set of all real numbers $\mathbb{R}$

Comparing $\mathbb{N}$ and set of all subsets of $\mathbb{N}$

Theorem (Cantor, 1891)

There is no bijection between $\mathbb{N}$ and the set of all subsets of $\mathbb{N}$.

Comparing $\mathbb{N}$ and set of all subsets of $\mathbb{N}$

Theorem (Cantor, 1891)

There is no bijection between $\mathbb{N}$ and the set of all subsets of $\mathbb{N}$.

- Proving existence just needs one to exhibit a function
- But how do we prove non-existence?

Comparing $\mathbb{N}$ and set of all subsets of $\mathbb{N}$

Theorem (Cantor, 1891)

There is no bijection between $\mathbb{N}$ and the set of all subsets of $\mathbb{N}$.

- Proving existence just needs one to exhibit a function
- But how do we prove non-existence? Try contradiction.

Comparing $\mathbb{N}$ and set of all subsets of $\mathbb{N}$

Theorem (Cantor, 1891)

There is no bijection between $\mathbb{N}$ and the set of all subsets of $\mathbb{N}$.

Proof by contradiction: Suppose there is such a bijection, say f. This would imply that each $i \in \mathbb{N}$ maps to some set $f(i) \subseteq \mathbb{N}$.

	0	1	2	3	...
$f(0)$	✓	×	×	×	...
$f(1)$	✓	×	✓	✓	...
$f(2)$	×	×	×	×	...
$f(3)$	×	✓	×	✓	...

Comparing $\mathbb{N}$ and set of all subsets of $\mathbb{N}$

Theorem (Cantor, 1891)

There is no bijection between $\mathbb{N}$ and the set of all subsets of $\mathbb{N}$.

Proof by contradiction: Suppose there is such a bijection, say f. This would imply that each $i \in \mathbb{N}$ maps to some set $f(i) \subseteq \mathbb{N}$.

	0	1	2	3	...
$f(0)$	~~✓~~ ×	×	×	×	...
$f(1)$	✓	~~×~~ ✓	✓	✓	...
$f(2)$	×	×	~~×~~ ✓	×	...
$f(3)$	×	✓	×	~~✓~~ ×	...

- Consider the set $S \subseteq \mathbb{N}$ obtained by switching the diagonal elements, i.e., $S = \{i \in \mathbb{N} \mid i \notin f(i)\}$.

Comparing $\mathbb{N}$ and set of all subsets of $\mathbb{N}$

Theorem (Cantor, 1891)

There is no bijection between $\mathbb{N}$ and the set of all subsets of $\mathbb{N}$.

Proof by contradiction: Suppose there is such a bijection, say f. This would imply that each $i \in \mathbb{N}$ maps to some set $f(i) \subseteq \mathbb{N}$.

	0	1	2	3	...
$f(0)$	$\not\checkmark$ $\times$	$\times$	$\times$	$\times$	...
$f(1)$	$\checkmark$	$\not\times$ $\checkmark$	$\checkmark$	$\checkmark$	...
$f(2)$	$\times$	$\times$	$\not\times$ $\checkmark$	$\times$	...
$f(3)$	$\times$	$\checkmark$	$\times$	$\not\checkmark$ $\times$	...

- Consider the set $S \subseteq \mathbb{N}$ obtained by switching the diagonal elements, i.e., $S = \{i \in \mathbb{N} \mid i \notin f(i)\}$.
- As f is bij, $\exists j \in \mathbb{N}, f(j) = S$.

Comparing $\mathbb{N}$ and set of all subsets of $\mathbb{N}$

Theorem (Cantor, 1891)

There is no bijection between $\mathbb{N}$ and the set of all subsets of $\mathbb{N}$.

Proof by contradiction: Suppose there is such a bijection, say f. This would imply that each $i \in \mathbb{N}$ maps to some set $f(i) \subseteq \mathbb{N}$.

	0	1	2	3	...
$f(0)$	$\not\checkmark$ $\times$	$\times$	$\times$	$\times$	...
$f(1)$	$\checkmark$	$\not\times$ $\checkmark$	$\checkmark$	$\checkmark$	...
$f(2)$	$\times$	$\times$	$\not\times$ $\checkmark$	$\times$	...
$f(3)$	$\times$	$\checkmark$	$\times$	$\not\checkmark$ $\times$	...

- Consider the set $S \subseteq \mathbb{N}$ obtained by switching the diagonal elements, i.e., $S = \{i \in \mathbb{N} \mid i \notin f(i)\}$.
- As f is bij, $\exists j \in \mathbb{N}, f(j) = S$.
- S and $f(j)$ differ at position j, for any j.

Comparing $\mathbb{N}$ and set of all subsets of $\mathbb{N}$

Theorem (Cantor, 1891)

There is no bijection between $\mathbb{N}$ and the set of all subsets of $\mathbb{N}$.

Proof by contradiction: Suppose there is such a bijection, say f. This would imply that each $i \in \mathbb{N}$ maps to some set $f(i) \subseteq \mathbb{N}$.

	0	1	2	3	$\ldots$
$f(0)$	$\not\checkmark$ $\times$	$\times$	$\times$	$\times$	$\ldots$
$f(1)$	$\checkmark$	$\not\times$ $\checkmark$	$\checkmark$	$\checkmark$	$\ldots$
$f(2)$	$\times$	$\times$	$\not\times$ $\checkmark$	$\times$	$\ldots$
$f(3)$	$\times$	$\checkmark$	$\times$	$\not\checkmark$ $\times$	$\ldots$

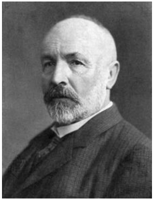
- Consider the set $S \subseteq \mathbb{N}$ obtained by switching the diagonal elements, i.e., $S = \{i \in \mathbb{N} \mid i \notin f(i)\}$.
- As f is bij, $\exists j \in \mathbb{N}, f(j) = S$.
- S and $f(j)$ differ at position j, for any j.
- Thus, $S \neq f(j)$ for all $j \in \mathbb{N}$, which is a contradiction! $\square$

Cantor's diagonalization

Does this proof look familiar??

Cantor's diagonalization

Does this proof look familiar??



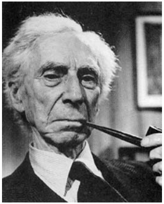


Figure: Cantor and Russell

Cantor's diagonalization

Does this proof look familiar??

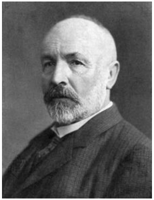
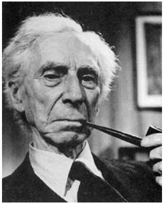



Figure: Cantor and Russell

- $S = \{i \in \mathbb{N} \mid i \notin f(i)\}$ is like the one from Russell's paradox.

Cantor's diagonalization

Does this proof look familiar??

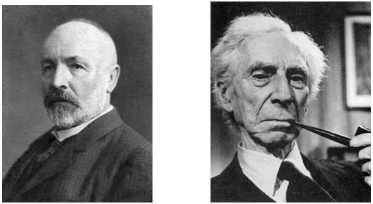


Figure: Cantor and Russell

- $S = \{i \in \mathbb{N} \mid i \notin f(i)\}$ is like the one from Russell's paradox.
- If $\exists j \in \mathbb{N}$ such that $f(j) = S$, then we have a contradiction.
  - If $j \in S$, then $j \notin f(j) = S$.
  - If $j \notin S$, then $j \notin f(j)$, which implies $j \in S$.

Cantor's diagonalization

Does this proof look familiar??

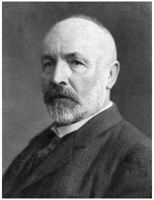
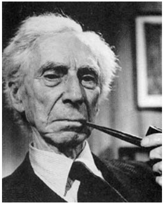



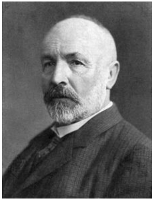
Figure: Cantor and Russell

In fact, using diagonalization Cantor showed that...

- There cannot be a bijection between any set and its power set (i.e., its set of subsets).(H.W)
- So there is an infinite hierarchy of "larger" infinities...

Cantor's diagonalization

Does this proof look familiar??



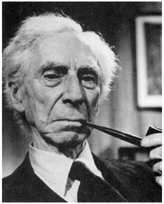


Figure: Cantor and Russell

In fact, using diagonalization Cantor showed that...

- There cannot be a bijection between any set and its power set (i.e., its set of subsets).(H.W)
- So there is an infinite hierarchy of "larger" infinities...
- There is no bijection from $\mathbb{R}$ to $\mathbb{N}$ (H.W). Moreover, there is a bijection from $\mathbb{R}$ to set of subsets of $\mathbb{N}$.