CS 105: Department Introductory Course on Discrete Structures

Instructor : S. Akshay

Aug 28, 2023

Lecture 09 – Basic Mathematical Structures

Uncountable Sets and relations

Countable and countably infinite sets

Definition

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- Proof 2 Show $f : P \to \mathbb{N}$ by f maps i^{th} prime to i is a bijection

Countable sets and functions

Are the following sets countable?

- the set of all integers $\mathbb{Z}$
- $\mathbb{N} \times \mathbb{N}$
- $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$
- the set of rationals $\mathbb{Q}$
- the set of all (finite and infinite) subsets of $\mathbb{N}$
- the set of all real numbers $\mathbb{R}$

Comparing $\mathbb{N}$ and set of all subsets of $\mathbb{N}$

Theorem (Cantor, 1891)

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Proof by contradiction: Suppose there is such a bijection, say f. This would imply that each $i \in \mathbb{N}$ maps to some set $f(i) \subseteq \mathbb{N}$.

	0	1	2	3	...
$f(0)$	✓	×	×	×	...
$f(1)$	✓	×	✓	✓	...
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- Consider the set $S \subseteq \mathbb{N}$ obtained by switching the diagonal elements, i.e., $S = \{i \in \mathbb{N} \mid i \notin f(i)\}$.

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- As f is bij, $\exists j \in \mathbb{N}, f(j) = S$.
- S and $f(j)$ differ at position j, for any j.
- Thus, $S \neq f(j)$ for all $j \in \mathbb{N}$, which is a contradiction! $\square$

Cantor's diagonalization

Does this proof look familiar??

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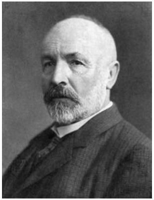

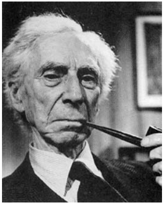

Figure: Cantor and Russell

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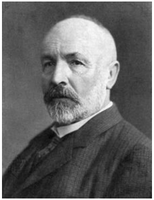 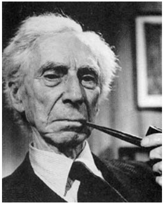

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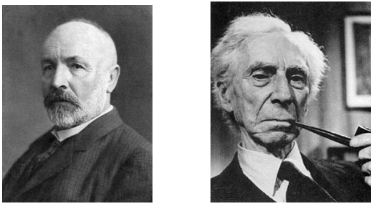

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- $S = \{i \in \mathbb{N} \mid i \notin f(i)\}$ is like the one from Russell's paradox.
- If $\exists j \in \mathbb{N}$ such that $f(j) = S$, then we have a contradiction.
  - If $j \in S$, then $j \notin f(j) = S$.
  - If $j \notin S$, then $j \notin f(j)$, which implies $j \in S$.

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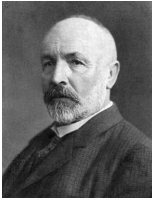 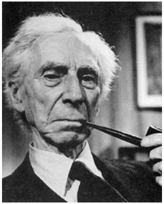

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- There cannot be a bijection between any set and its power set (i.e., its set of subsets).(H.W)
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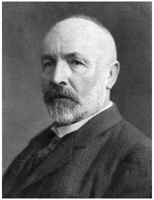 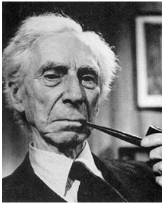

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In fact, using diagonalization Cantor showed that...

- There cannot be a bijection between any set and its power set (i.e., its set of subsets).(H.W)
- So there is an infinite hierarchy of "larger" infinities...
- There is no bijection from $\mathbb{R}$ to $\mathbb{N}$ (H.W). Moreover, there is a bijection from $\mathbb{R}$ to set of subsets of $\mathbb{N}$.

One infinity is “strictly” bigger than another!

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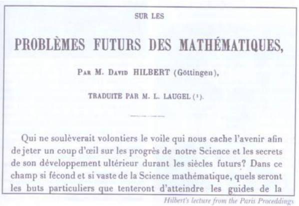
Cantor’s Continuum hypothesis

There is no set whose “cardinality” is strictly between $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$ (i.e., between naturals and reals).



SUR LES

PROBLÈMES FUTURS DES MATHÉMATIQUES,

PAR M. DAVID HILBERT (Göttingen),

TRADUITE PAR M. L. LAUGEL (1).

Qui ne soulèverait volontiers le voile qui nous cache l’avenir afin de jeter un coup d’œil sur les progrès de notre Science et les secrets de son développement ultérieur durant les siècles futurs? Dans ce champ si fécond et si vaste de la Science mathématique, quels seront les buts particuliers que tenteront d’atteindre les guides de la

Hilbert’s lecture from the Paris Proceddings

Figure: 1st of Hilbert’s 23 problems for the 20th century in 1900.

What did the world think about these proofs (in 1890s?)

(a) Kronecker

(b) Poincare

(c) Theologians

- Kronecker: Only constructive proofs are proofs! "Scientific Charlatan", "Corruptor of youth"!
- Poincare: Set theory is a "disease" from which mathematics will be cured.
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- Hilbert: No one can expel us from the paradise that Cantor has created for us.

Summary and moving on...

- Finite and infinite sets.
- Using functions to compare sets: focus on bijections.
- Countable, countably infinite and uncountable sets.
- Cantor's diagonalization argument (A new powerful proof technique!).

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Next: Basic Mathematical Structures – Relations

Relations

Definition: Function

Let A, B be two sets. A function f from A to B is a subset R of $A \times B$ such that

(i) $\forall a \in A, \exists b \in B$ such that $(a, b) \in R$, and

(ii) if $(a, b) \in R$ and $(a, c) \in R$, then $b = c$.

- Now, suppose A is the set of all Btech students and B is the set of all courses. Clearly, we can assign to each student the set of courses he/she is taking. Is this a function?

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- By removing the two extra assumptions in the defn, we get:

Definition: Relation

- A relation R from A to B is a subset of $A \times B$. If $(a, b) \in R$, we also write this as $a \; R \; b$.
- Thus, a relation is a way to relate the elements of two (not necessarily different) sets.

Examples and representations of relations

We write $R(A, B)$ for a relation from A to B and just $R(A)$ if $A = B$. Also if A is clear from context, we just write R.

Examples of relations

- All functions are relations.
- $R_1(\mathbb{Z}) = \{(a, b) \mid a, b \in \mathbb{Z}, a - b \text{ is even } \}$.
- $R_2(\mathbb{Z}) = \{(a, b) \mid a, b \in \mathbb{Z}, a \leq b\}$.
- Let S be a set, $R_3(\mathcal{P}(S)) = \{(A, B) \mid A, B \subseteq S, A \subseteq B\}$.
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Representations of a relation from A to B.

- As a set of ordered pairs of elements, i.e., subset of $A \times B$.
- As a directed graph.
- As a (database) table.

Use of relations

Practical application in relational databases: IMDB, university records, etc.

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  - Equivalence relations
  - Partial orders

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- if $S' \in P$, then $S' \neq \emptyset$.
- $\bigcup_{S' \in P} S' = S$: its union covers entire set S.
- If $S_1, S_2 \in P$, then $S_1 \cap S_2 = \emptyset$: sets are disjoint.

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Can you think of two trivial partitions that any set must have?

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What properties does this relation have?