CS 105: Department Introductory Course on Discrete Structures

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Lecture 11 – Basic Mathematical Structures

Equivalence relations and partially ordered sets

Recap: Proofs and Structures

Chapter 1: Proofs

1. Propositions, predicates
2. Types of proofs, axioms
3. Mathematical Induction, Well-ordering principle
4. Strong Induction

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Chapter 2: Sets and Functions

1. Finite and infinite sets.
2. Using functions to compare sets: focus on bijections.
3. Countable, countably infinite and uncountable sets.
4. Cantor's diagonalization (New/powerful proof technique!).

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Chapter 3: Relations

1. Equivalence Relations
2. Partial Orders

Examples

- Reflexive: $\forall a \in S$, aRa.
- Symmetric: $\forall a, b \in S$, aRb implies bRa.
- Transitive: $\forall a, b, c \in S$, aRb, bRc implies aRc.
- Equivalence: Reflexive, Symmetric and Transitive.

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Relation	Refl.	Sym.	Trans.	Equiv.
aR_4b if students a and b take same set of courses	✓	✓	✓	✓
aR_5b if student a takes course b				
$\{(a,b) \mid a,b \in \mathbb{Z}, (a-b) \mod 2 = 0\}$				
$\{(a,b) \mid a,b \in \mathbb{Z}, a \leq b\}$				
$\{(a,b) \mid a,b \in \mathbb{Z}, a < b\}$				
$\{(a,b) \mid a,b \in \mathbb{Z}, a \mid b\}$				
$\{(a,b) \mid a,b \in \mathbb{R}, \lvert a-b \rvert < 1\}$				
$\{((a,b),(c,d)) \mid (a,b),(c,d) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}), (ad = bc)\}$				

Equivalence classes

Definition

- Let R be an equivalence relation on set S, and let $a \in S$.
- Then the equivalence class of a, denoted $[a]$, is the set of all elements related to it, i.e., $[a] = \{b \in S \mid (a, b) \in R\}$.

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Proof Sketch: (1) to (2) symm and trans, (2) to (3) refl, (3) to (1) symm and trans. (H.W.: Redo the proof formally.)

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Proof sketch of (1): Union, non-emptiness follows from reflexivity. The rest (pairwise disjointness) follows from the previous lemma.

(H.W.): Write the formal proofs of (1) and (2).

More “applications” of equivalence relations

Defining new objects using equivalence relations

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- Then the equivalence classes of R define the rational numbers.
- e.g., $\left[\frac{1}{2}\right] = \left[\frac{2}{4}\right]$ are two names for the same rational number.
- Indeed, when we write $\frac{p}{q}$ we implicitly mean $\left[\frac{p}{q}\right]$.

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Can we define integers and real numbers starting from naturals by using equivalence classes?

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Examples:

- $R_1(\mathbb{Z}) = \{(a,b) \mid a,b \in \mathbb{Z}, a \leq b\}$.
- $R_2(\mathcal{P}(S)) = \{(A,B) \mid A,B \in \mathcal{P}(S), A \subseteq B\}$.

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Definition

A partial order is a relation which is reflexive, transitive and anti-symmetric.

Partial orders and equivalences relations

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- Anti-symmetric: $\forall a, b \in S$, aRb, bRa implies $a = b$.
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	Reflexive	Transitive	Symmetric	Anti-symmetric
Equivalence relation	✓	✓	✓	
Partial order	✓	✓		✓

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	Refl.	Anti-Sym	Trans.	PO
$\{(a,b) \mid a,b \in \mathbb{Z}, a \leq b\}$	✓	✓	✓	✓
$\{(A,B) \mid A,B \in \mathcal{P}(S), A \subseteq B\}$	✓	✓	✓	✓
$\{(a,b) \mid a,b \in \mathbb{Z}, a < b\}$				
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  - i.e., $\forall a, b \in S$, either $a \preceq b$ or $b \preceq a$.

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- A total order is a partial order $\preceq$ on S in which every pair of elements is comparable
- Qn: Can a relation be symmetric and anti-symmetric?
- Qn: Can a relation be neither symmetric nor anti-symmetric?

Partially ordered sets (Posets)

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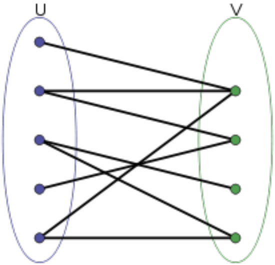
Examples

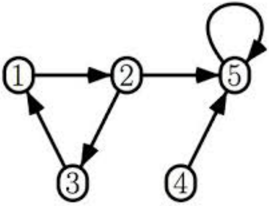
- $(\mathbb{Z}, \leq)$: integers with the usual less than or equal to relation.
- $(\mathcal{P}(S), \subseteq)$: powerset of any set with the subset relation.
- $(\mathbb{Z}^+, \mid\,)$: positive integers with divisibility relation.

Graphical representation of relations: posets

Recall: any relation on a set can be represented as a graph with

- nodes as elements of the set and
- directed edges between them indicating the ordered pairs that are related.





- Did these come from posets?
- Do graphs defined by posets have any "special" properties?

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- Let $S = \{1, 2, 3\}$. Recall the poset $(\mathcal{P}(S), \subseteq)$.
- How does the graph of $(\mathcal{P}(S), \subseteq)$ look like?

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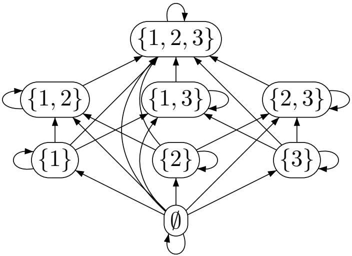

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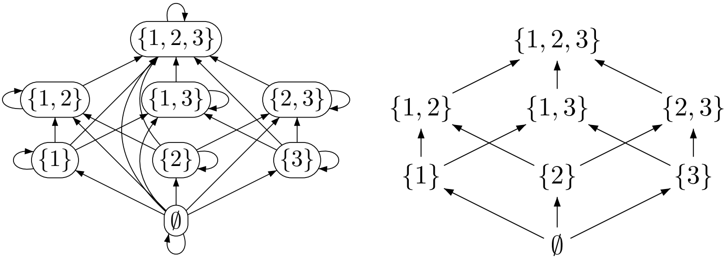

Figure: Graph of a poset and its Hasse diagram

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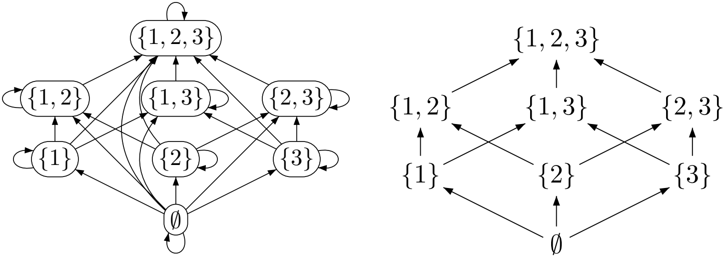

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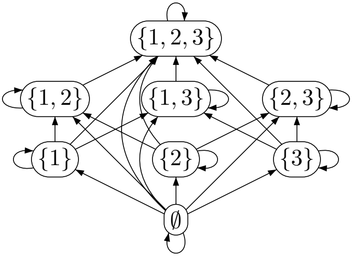

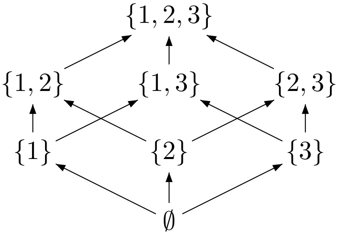

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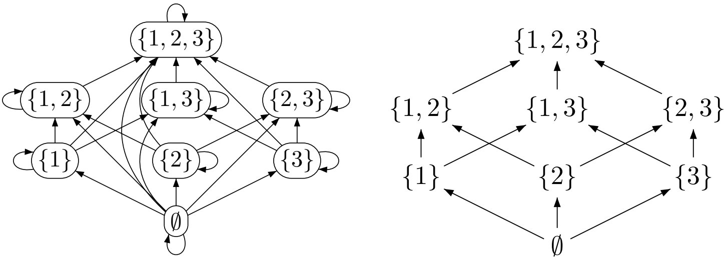

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- Starting from a node and following the directed edges (except self-loops), one can’t come back to the same node.
- Given the Hasse diagram of a poset, its reflexive transitive closure gives back the graph of the poset.