CS 105: Department Introductory Course on Discrete Structures

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Guest Lecture by : R. Govind

Sep 04, 2023
Lecture 12 – Basic Mathematical Structures
Chains and Antichains

Recap: Partial order relations

Last class we saw

- Partial orders: definition and examples
- Posets
- Graphical representation as Directed Acyclic Graphs

Definition

- A partial order is a relation which is reflexive, transitive and anti-symmetric.
- A total order is a partial order in which every pair of elements is comparable.

- A poset is a set S with a partial order $\preceq \;\subseteq S \times S$.

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Examples

- $(\mathbb{Z}, \leq)$: integers with the usual less than or equal to relation.
- $(\mathcal{P}(S), \subseteq)$: powerset of any set with the subset relation.
- $(\mathbb{Z}^+, \mid)$: positive integers with divisibility relation.

Recap: Partial order relations

Let $S = \{1, 2, 3\}$. Recall the poset $(\mathcal{P}(S), \subseteq)$. How does the graph of $(\mathcal{P}(S), \subseteq)$ look like?

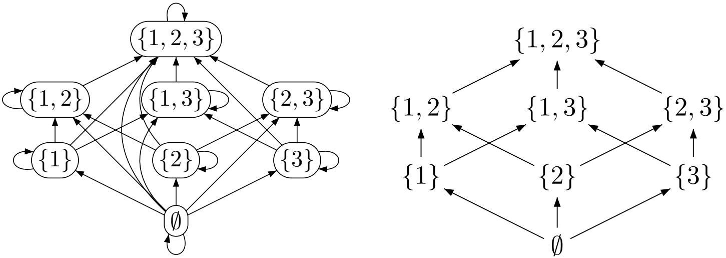

Figure: Graph of a poset and its Hasse diagram

Minimal and maximal elements

Let $(S, \preceq)$ be a poset.

- An element $a \in S$ is called minimal if, $b \preceq a$ implies $b = a$.
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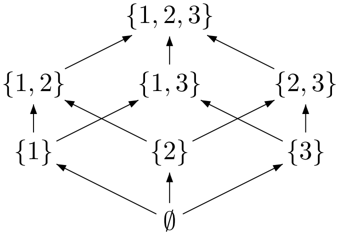

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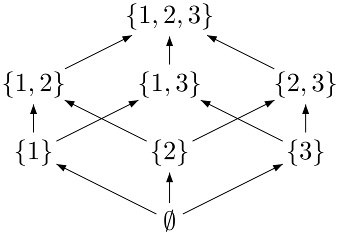

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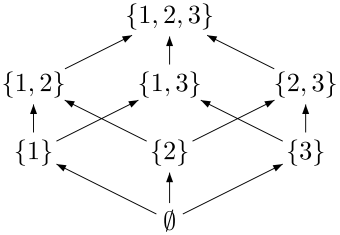

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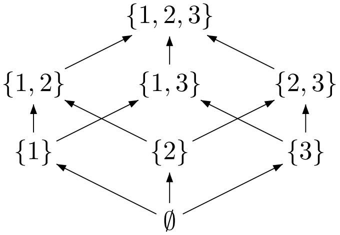

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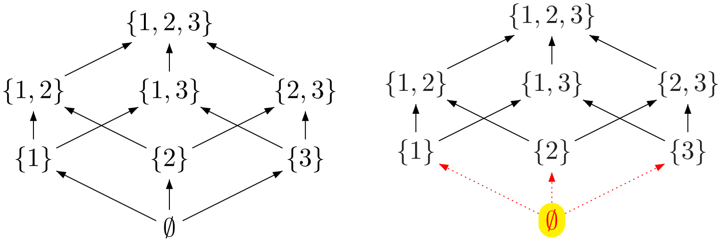

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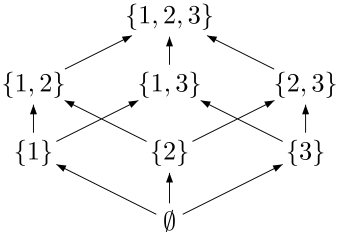

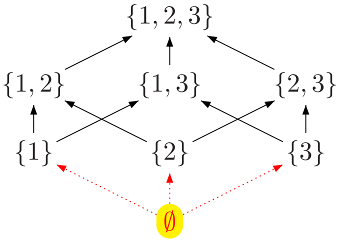

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- What are the minimal element(s?) in $(\mathbb{Z}_{>1}, |)$.

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  Contradiction!

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Proof by induction?(H.W)

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What about infinite posets?

Posets: Chains and Antichains

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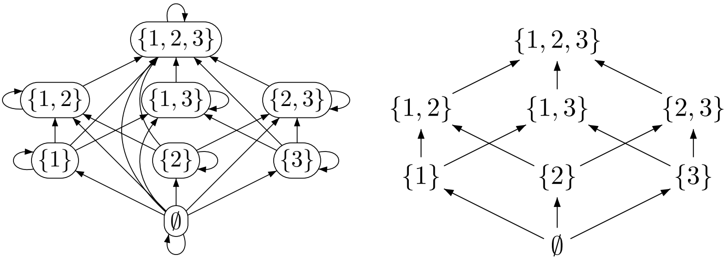

Figure: Graph of a poset and its Hasse diagram

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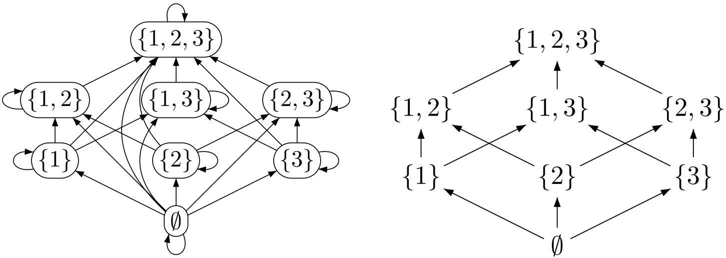

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- Subsets that are totally ordered?
- Subsets that are unordered?

Chains and Anti-chains

Definition

Let $(S, \preceq)$ be a poset. A subset $B \subseteq S$ is called

- a chain if every pair of elements in B is related by $\preceq$.
- That is, $\forall a, b \in B$, we have $a \preceq b$ or $b \preceq a$ (or both).

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- That is, $\forall a, b \in A, a \neq b$, we have neither $a \preceq b$ nor $b \preceq a$.

Chains and Anti-chains: examples

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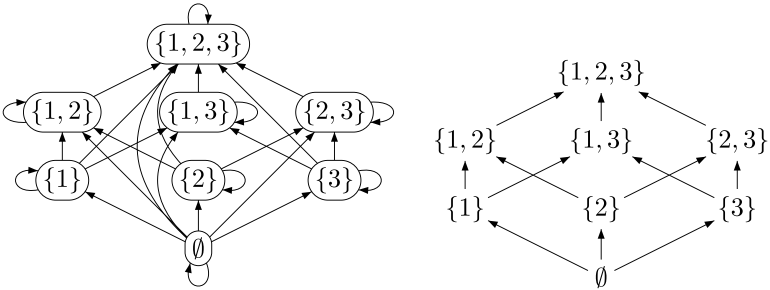

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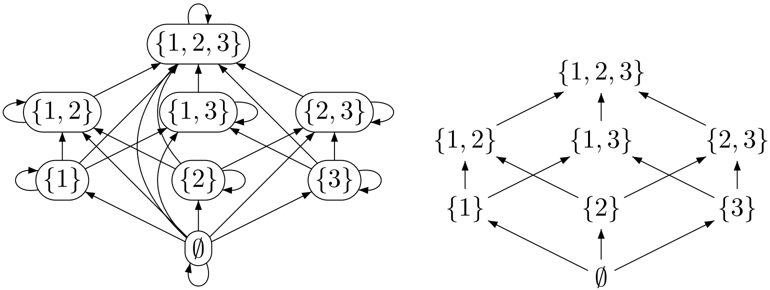

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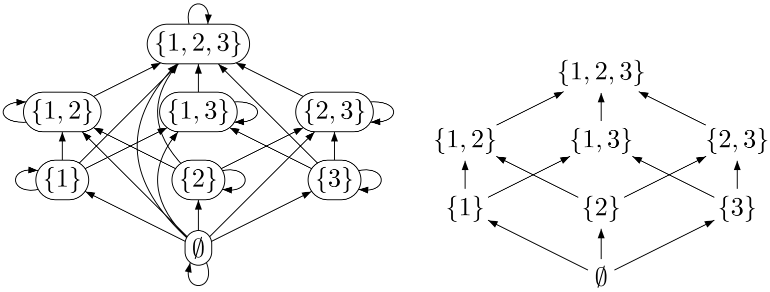

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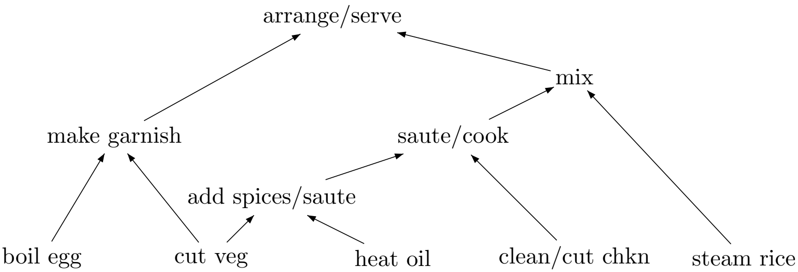
- What are the chains in this poset?
- What are the anti-chains in this poset?
- Give an example of an infinite chain & anti-chain in $(\mathbb{Z}^+, |)$.

Examples and applications

A task scheduling example

Let us represent a recipe for making Chicken Biriyani as a poset!



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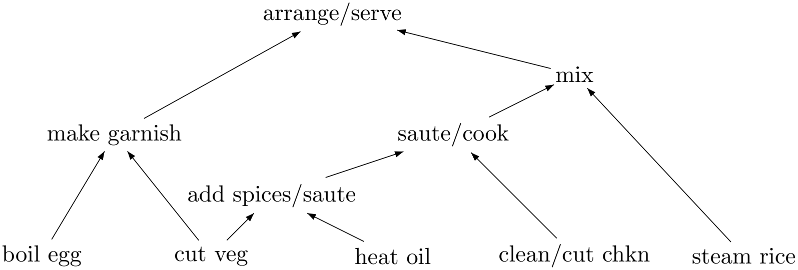

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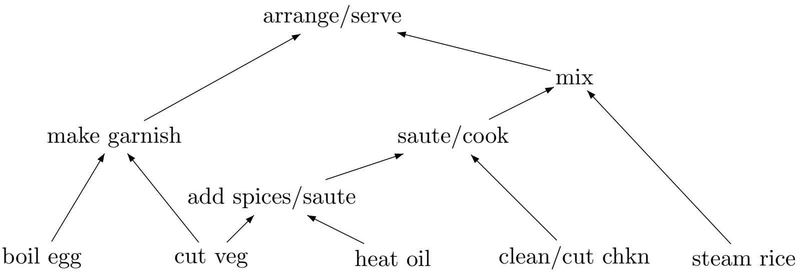

- Clearly, this shows the dependencies.
- But when you cook you need a total order, right?
- Further, this total order must be consistent with the po.
- This is called a linearization or a topological sorting.

Topological sorting

Definition

A topological sort or a linearization of a poset $(S, \preceq)$ is a poset $(S, \preceq_t)$ with a total order $\preceq_t$ such that $x \preceq y$ implies $x \preceq_t y$.

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Proof: (H.W)

- Recall the lemma:
  - Every finite non-empty poset has at least one minimal element (x is minimal if $\nexists y, y \preceq x$).
- Then, construct a (new) chain to complete the proof.