# CS 105: DIC on Discrete Structures

Instructor : S. Akshay

Sept 11, 2023
Lecture 15 – Counting

# Least upper bounds and greatest lower bounds

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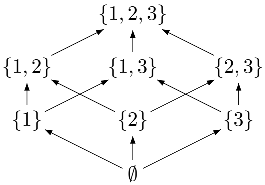


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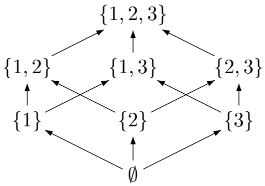


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- Let $A = \{\{1\}, \{2\}\}$. Then $\{1, 2\}, \{1, 2, 3\}$ are upper bounds of $A$ in $P_1$ and $\{1, 2\}$ is the lub of $A$.

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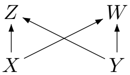


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Poset $P_3 = (S_3, \preceq)$

- Consider $P_3 = (S_3, \preceq)$ where $S_3 = \{X, Y, Z, W\}$ and the $\preceq$ is as given by the arrows. Let $B = \{X, Y\}$. Then $Z, W$ are both upper bounds of $B$ in $P_3$, but $B$ has no lub in $P_3$.

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**Some Obervations (Exercise: Prove it!)**

- The lub/glb of a subset $A$ in $S$, if it exists, is unique.
- If the lub/glb of $A \subseteq S$ belongs to $A$, then it is the greatest/least element of $A$.

# Lattices

## Definition

- A lattice is a poset in which every pair of elements has both a lub and a glb (in the set), i.e., $\forall x, y \in S$, there exists $l, u \in S$ such that $l$ is the glb and $u$ is the lub of $\{x, y\}$.

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- What about $(\{2, 4, 5, 10, 12, 20, 25\}, |)$ ?

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## Applications of Lattices

- Models of information flow – think security clearence.
- Finite lattices have a strong link with Boolean Algebra
- Several other applications in many domains of mathematics and CS, including formal semantics of programming languages, program verification.

# Next chapter: Counting and Combinatorics

**Topics to be covered**

- Basics of counting
- Subsets, partitions, Permutations and combinations
- Recurrence relations and generating functions
- Pigeonhole Principle and its extensions

# Introduction to combinatorics

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- Enumerative combinatorics: counting combinatorial/discrete objects e.g., sets, numbers, structures...
- Existential combinatorics: show that there exist some combinatorial "configurations".
- Constructive combinatorics: construct interesting configurations...

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**The product principle**

If there are $n_1$ ways of doing something and $n_2$ ways of doing another thing, then there are $n_1 \cdot n_2$ ways of performing both actions.

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  - Product principle: two choices for each element, hence $2 \cdot 2 \cdots 2 \cdot 2$ ($n$-times).
  - Bijection: between $\mathcal{P}(X)$ and $n$-length sequences over $\{0, 1\}$ (characteristic vector).
  - Induction: Since we already know the answer!
  - Recurrence: $F(n) = 2 \cdot F(n-1), F(0) = 1$. solve it?
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**Sum Principle**

If something can be done in $n_1$ or $n_2$ ways such that none of the $n_1$ ways is the same as any of the $n_2$ ways, then the total number of ways to do this is $n_1 + n_2$.

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- But, how many subsets of size $k$ does a set of $n$ elements have? This number, denoted $\binom{n}{k}$, is called a binomial coefficient.
- We all know(?) that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Prove it!

# Permutations and combinations

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- Equate them! Principle of double counting.
  - if you can't count something, count something else and count it twice over!

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**Permutations and combinations**

- No. of $k$-size subsets of set of size $n$ = No. of $k$-combinations of a set of $n$ (distinct) elements = $\binom{n}{k}$.
- No. of $k$-size ordered subsets of set of size $n$ = No. of $k$-permutations of a set of $n$ (distinct) elements.

# Simple examples to illustrate “double counting”

**Prove the following identities (by only using double counting!)**

1. $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n.$
2. $\displaystyle\binom{n}{k} = \binom{n}{n-k}.$
3. $\displaystyle k\binom{n}{k} = n\binom{n-1}{k-1}$
4. $\displaystyle\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

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The latter two are in fact recursive definitions for $\binom{n}{k}$. What are the boundary conditions?

# A more interesting example with double counting

**Handshake Lemma**

At a meeting with $n$ people, the number of people who shake hands an odd number of times is even.

What will you count here?

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Proof in six steps:

1. Define a relation $R$: $iRj$ if $i$ and $j$ shook hands.
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5. But now, let $X$ be the total number of handshakes. Clearly this is an integer. Total no. of directed edges $= 2 \cdot X$.
6. This implies, $\sum_i m_i = 2 \cdot X$. Which means that number of $i$ such that $m_i$ is odd is even!