CS 105: DIC on Discrete Structures

Instructor : S. Akshay

Sept 16, 2023

Lecture 16 – Counting and Combinatorics

Counting and Combinatorics

Topics to be covered

- Basics of counting
  - Product principle
  - Sum principle
  - Bijection principle
  - Double counting

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  1. Binomial coefficients and Binomial theorem
  2. Pascal's triangle
  3. Permutations and combinations with repetitions

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  1. Binomial coefficients and Binomial theorem
  2. Pascal's triangle
  3. Permutations and combinations with repetitions
- Recurrence relations and generating functions
- Pigeonhole Principle and its extensions

Recall: interesting example with double counting

Handshake Lemma

At a meeting with n people, the number of people who shake hands an odd number of times is even.

What will you count here?

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Proof in six steps:

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4. Therefore, Total no. of directed edges $= \sum_i m_i$.
5. But now, let X be the total number of handshakes. Clearly this is an integer. Total no. of directed edges $= 2 \cdot X$.
6. This implies, $\sum_i m_i = 2 \cdot X$. Which means that number of i such that m_i is odd is even!

Binomial theorem

Recall: $\sum_{k=0}^{n} \binom{n}{k} = 2^n.$

We generalize this...

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We generalize this...

Binomial Theorem

Let x, y be variables and $n \in \mathbb{Z}^{\geq 0}$. Then,

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$$(x+y)^1 = x + y$$

$$(x+y)^2 = (x+y)(x+y) = x^2 + 2xy + y^2$$

$$(x+y)^3 = (x+y)(x+y)^2 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = (x+y)(x+y)^3 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

...

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(H.W-1) Prove this by induction.

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Proof (combinatorial):

1. Consider any term $x^i y^j$, where $i + j = n$.
2. To get $x^i y^j$ term in

$$(x+y)(x+y)\cdots(x+y) \qquad (n \text{ times})$$

we need to pick j $y's$ from n sums and remaining x's.

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   we need to pick j $y's$ from n sums and remaining x's.
3. Thus, the coefficient of this term = number of ways to get this term = number of ways to pick j y's from n elts = $\binom{n}{j}$.

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Corollaries:

1. $\binom{n}{j} = \binom{n}{n-j}$,
2. $\displaystyle\sum_{j=0}^{n} \binom{n}{j} 2^j = 3^n$.
3. No. of subsets of n-element set having even cardinality = No. of subsets of n-element set having odd cardinality ? (H.W-2)

Pascal's Triangle

A recursive way to compute binomial coefficients

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$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

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1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Fun with Pascal's triangle

1

1 1

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1 3 3 1

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1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

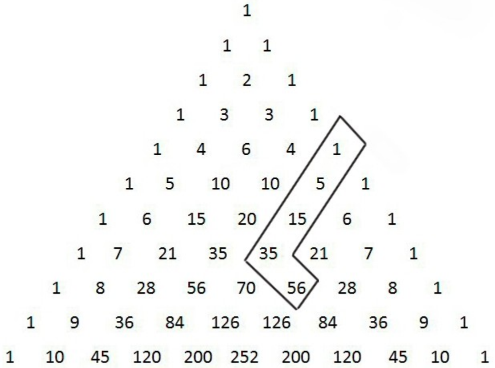
1 9 36 84 126 126 84 36 9 1

1 10 45 120 200 252 200 120 45 10 1

Some simple observations. Recall: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

1. Row i adds up to 2^i, Row $i+1$ adds up to twice of row i.
2. Sequence of numbers, squares, cubes?

Fun with Pascal's triangle



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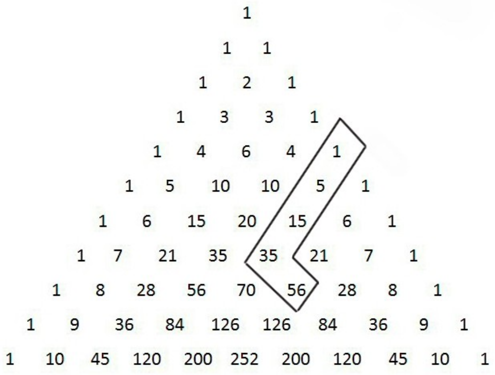

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1. Row i adds up to 2^i, Row $i+1$ adds up to twice of row i.
2. Sequence of numbers, squares, cubes?
3. Hockey stick patterns: (H.W-3)
   $\binom{n+1}{m} = \binom{n}{m} + \binom{n-1}{m-1} \ldots + \binom{n-m}{0}$

Fun with Pascal's triangle

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- For some rows, all values in the row (except first and last) are divisible by the second!

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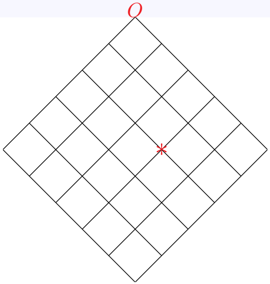
- In fact, for all prime rows? why should p divide $\binom{p}{r}$, $r < p$?
- Corollary: $2^p - 2$ is a multiple of p, for any prime p.
- Interesting Ex.: Count no. of odd numbers in each row...

An application to path counting

Map problems

From the top corner, how many shortest routes lead to a particular junction?



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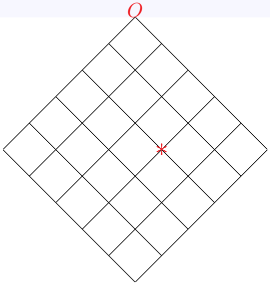

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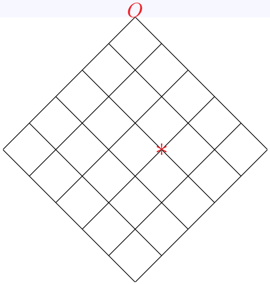

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- Our path has 5 segments of which there must be 3 R's.

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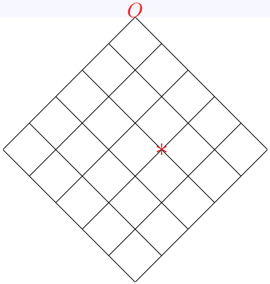

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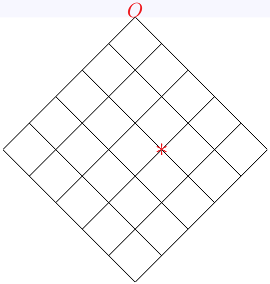

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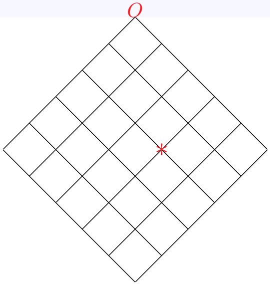

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H.W-4: Prove/verify this formally.

Permutations and Combinations with repetitions

How many ways can you select k objects from a set of n elements?

- Depends on whether order is significant: If yes permutations, else combinations.
- What if repetitions are allowed?

	Order significant	Order not significant
Repetitions not allowed	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$
Repetitions are allowed	n^k	??

Combinations with repetitions

Theorem

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3. There is a bijection between such lists and k-sets of n-elements with repetitions allowed (why?).

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4. Thus, question reduces to no. of ways to choose k stars or $n-1$ bars from a set of $n-k+1$ positions $= \binom{n+k-1}{k}$. $\square$

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- H.W-5: How many solutions does the equation $x_1 + x_2 + x_3 + x_4 = 17$ have such that $x_1, x_2, x_3, x_4 \in \mathbb{Z}^{\geq 0}$?