CS 105: DIC on Discrete Structures

Instructor : S. Akshay

Sept 14, 2023
Lecture 17 – Counting and Combinatorics
Recurrence Relations

Logistics: Midsem

CS105 Midsem Exam

- Date and Time: Sept 23rd, Saturday, at 16:00 hrs, 4pm

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Syllabus

First 16 lectures of course, till & incl Tuesday Sept 12.

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Syllabus

First 16 lectures of course, till & incl Tuesday Sept 12.

- Propositions, proofs, induction,
- Basic structures: sets, functions, (un)countability, relations, posets, chains, anti-chains, lattices.
- Basic counting: counting principles, double counting, permutations & combinations.

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- Only Basic material, NOT advanced.
- Please follow piazza and ask, if necessary.

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- Solve more questions from Kenneth Rosen etc.
- Few more extra/advanced questions may be released (no solutions, but can discuss on piazza).

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- Basics of counting
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  - Sum principle
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  - Sum principle
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  - Double counting
- Subsets, partitions, Permutations and combinations
  1. Binomial coefficients and Binomial theorem
  2. Pascal's triangle
  3. Permutations and combinations with repetitions

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How big is $n!$?

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where e is the base of natural logarithms, $\log(e) = e^{\log(e)} = 1$.

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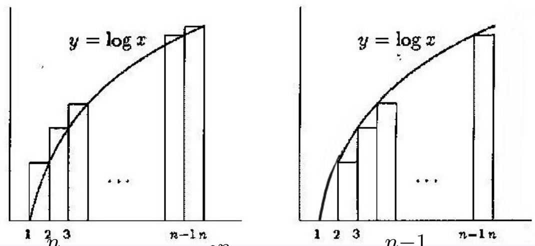
Now, we relate it to natural log function as shown in the figure.



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- r.h.s. $n! \le e^{(n+1)\log(n)-n+1} = n^{n+1}/e^{n-1} = ne(n/e)^n$. □

Next: Recurrence relations and generating functions

Recall: No. of subsets of a set of n elements

How many subsets does a set A of n elements have?

- Induction
- Product principle: two choices for each element, hence $2 \cdot 2 \cdots 2 \cdot 2$ (n-times).
- Bijection: between $\mathcal{P}(X)$ and n-length $\{0,1\}$-sequences.
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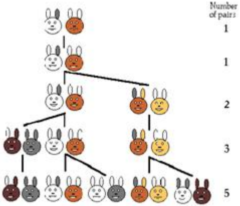
But how do you solve it?

Another example of recurrence: The Fibonacci Sequence



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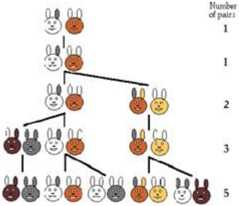

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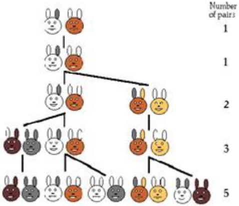

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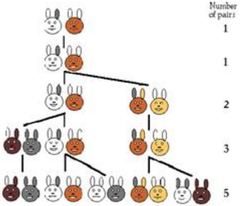

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- Consider $u_n = u_{n-1} + u_{n-2} - u_{n-3}$ where $u_2 = 2, u_1 = u_0 = 1$

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- A recurrence relation for a sequence is an equation that expresses its n^{th} term using one or more of the previous terms of the sequence.
- A linear recurrence relation is of the form

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- k is called the degree/depth of the sequence.
- The first few (e.g., k elements $u_0, \ldots, u_{k-1}$) are initial conditions and they determine the whole sequence.

Some more examples of recurrences

How many bit strings of length n are there that do not have two consecutive 0's?

- Find a recurrence relation for this
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  In general, let $C(n)$ be the number of ways of doing this.

An aside: find the Fibonacci sequence!

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 200 252 200 120 45 10 1

- $F(n) = F(n-1) + F(n-2)$.
- $1, 1, 2, 3, 5, 8, 13, \ldots$.
- Can you observe the sum of which terms in the Pascal's triangle gives rise to the terms of the Fibonacci sequence?