CS 105: DIC on Discrete Structures

Instructor : S. Akshay

Sept 26, 2023
Lecture 19 – Counting and Combinatorics
Solving Recurrence relations via generating functions

Last few weeks

Basic counting techniques and applications

1. Sum and product, bijection, double counting principles
2. Binomial coefficients and binomial theorem, Pascal's triangle
3. Permutations and combinations with/without repetitions
4. Counting subsets, relations, Handshake lemma
5. Stirling's approximation: Estimating $n!$
6. Recurrence relations and one method to solve them.

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Today

Solving recurrence relations via generating functions.

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- Recall the recurrence for Catalan Numbers:
  $C(n) = \sum_{i=1}^{n-1} C(i)C(n-i)$ for $n > 1$, $C(0) = C(1) = 1$.
  No. of ways to bracket a sum of n terms s.t. it can be computed by adding two numbers at a time?

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Solving recurrence relations

By solving, we mean give a closed-form expression for n^{th} term.

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2. So if $\alpha^2 - \alpha - 1 = 0$, the recurrence holds for all n.
3. Solving, $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$
4. Thus, general solution is $F_n = a(\frac{1+\sqrt{5}}{2})^n + b(\frac{1-\sqrt{5}}{2})^n$.
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(t+t^2)\phi(t) &= \sum_{n=0}^{\infty} F(n) t^n - 1 \\
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thus, as before

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Properties of generating functions

Definition

The (ordinary) generating function for a sequence $a_0, a_1, \ldots \in \mathbb{R}$ is the infinite series $\phi(x) = \sum_{k=0}^{\infty} a_k x^k$.

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  1. If $f(x) = g(x)$, then $a_k = b_k$ for all k.
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If you don't like this, take $x \in \mathbb{R}, |x| < 1$.

Simple examples using generating functions

Standard identities:

- $\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k$
- $\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk}$
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2. Solve the recurrence $a_k = 8a_{k-1} + 10^{k-1}$ with $a_0 = 1, a_1 = 9$.

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  - Compare coefficients of x^n in $(1 + x)^{2n} = ((1 + x)^n)^2$.
- (H.W) Write a recurrence for the number of derrangements. That is, no. of ways to arrange n letters into n addressed envelopes such that no letter goes to the correct envelope.

Other applications of generating functions

- What is the number of ways a_k of selecting k elements from an n element set if repetitions are allowed?
  - Let $\phi(x) = \sum_{k=0}^{\infty} a_k x^k$.
  - Observe that $\phi(x) = (1 + x + x^2 + \dots)^n = (1 - x)^{-n}$
  - Expand this by the extended binomial theorem and compare coefficients of x^k.
  - $a_k = \binom{-n}{k}(-1)^k = (-1)^k \binom{n+k-1}{k}(-1)^k = \binom{n+k-1}{k}$.
  - (H.W) What if there must be ≥ 1 element of each type?
- Proving binomial identities: Show that $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.
  - Compare coefficients of x^n in $(1+x)^{2n} = ((1+x)^n)^2$.
- (H.W) Write a recurrence for the number of derrangements. That is, no. of ways to arrange n letters into n addressed envelopes such that no letter goes to the correct envelope.
- (H.W) How many ways can a convex n-sided polygon be cut into triangles by adding non-intersecting diagonals (i.e., connecting vertices with non-crossing lines)? Write a recurrence and solve it!