CS 105: DIC on Discrete Structures

Instructor : S. Akshay

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Lecture 20 – Counting and Combinatorics

Some Applications of Generating functions, Principle of Inclusion-Exclusion

Last few weeks

Basic counting techniques and applications

1. Sum and product, bijection, double counting principles
2. Binomial coefficients and binomial theorem, Pascal's triangle
3. Permutations and combinations with/without repetitions
4. Counting subsets, relations, Handshake lemma
5. Stirling's approximation: Estimating $n!$
6. Recurrence relations and one method to solve them.
7. Solving recurrence relations via generating functions.

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Reading assignment

Read examples/generalizations from Sections 6.1 and 6.2 from Rosen's book (6th Indian Edition). In International 7th version its Sec 8.2 and 8.4?

Properties of generating functions

Definition

The (ordinary) generating function for a sequence $a_0, a_1, \ldots \in \mathbb{R}$ is the infinite series $\phi(x) = \sum_{k=0}^{\infty} a_k x^k$.

- Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then
  1. If $f(x) = g(x)$, then $a_k = b_k$ for all k.
  2. $f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$,
  3. $f(x)g(x) = \sum_{k=0}^{\infty} (\sum_{j=0}^{k} a_j b_{k-j}) x^k$,
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- Let $u \in \mathbb{R}$, $k \in \mathbb{Z}^{\geq 0}$, Then extended binomial coefficient $\binom{u}{k}$ is defined as $\binom{u}{k} = \frac{u(u-1)\ldots(u-k+1)}{k!}$ if $k > 0$ and $= 1$ if $k = 0$.
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If you don't like this, take $x \in \mathbb{R}, |x| < 1$.

Simple examples using generating functions

Standard identities:

- $\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k$
- $\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk}$
- $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

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- (H.W) Write a recurrence for the number of derrangements. That is, no. of ways to arrange n letters into n addressed envelopes such that no letter goes to the correct envelope.
- (H.W) How many ways can a convex n-sided polygon be cut into triangles by adding non-intersecting diagonals (i.e., connecting vertices with non-crossing lines)? Write a recurrence and solve it!

Solving Catalan numbers using generating functions

Catalan Numbers

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- Now consider $\phi(x)^2$.
- $\phi(x)^2 = (\sum_{k=1}^{\infty} C(k)x^k)(\sum_{k=1}^{\infty} C(k)x^k)$
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- Solving for $\phi(x)$ we get, $\phi(x) = \frac{1}{2}(1 \pm (1-4x)^{1/2})$
- But since $\phi(0) = 0$, we have
  $\phi(x) = \frac{1}{2}(1 - (1-4x)^{1/2}) = \frac{1}{2} + (-\frac{1}{2}(1-4x)^{1/2})$.

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Recall: Extended binomial theorem

Let $\alpha \in \mathbb{R}$, $(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$, where $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}$.

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- $C(k) = \frac{1 \cdot 4^k}{2^{k+1} \cdot k!} \cdot \frac{1.2....(2k-3)(2k-2)}{2^{k-1}(k-1)!} = \frac{(2k-2)!}{k!(k-1)!}$.

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Thus, the n^{th} Catalan number is given by

$C(n) = \frac{(2n-2)!}{n!(n-1)!} = \frac{1}{n}\binom{2n-2}{n-1}$

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A simple example:

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Theorem: Principle of Inclusion-Exclusion (PIE)

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then,

$$|A_1 \cup \ldots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1} |A_1 \cap \ldots \cap A_n|$$

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- $|A_i| = (m-1)^n$, $|A_i \cap A_j| = (m-2)^n$...

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- But now what is $|A_i|, |A_i \cap A_j|, |A_i \cap A_j \cap A_k|, \ldots$?
- $|A_i| = (m-1)^n$, $|A_i \cap A_j| = (m-2)^n$...
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Number of surjections

- How many surjections are there from $[n] = \{1, \ldots, n\}$ to $[m] = \{1, \ldots, m\}$?
- # surjections = total #functions - those that miss some element in range.
- Let $A_i = \{f : [n] \to [m] \mid i \notin Range(f)\}$
- Then, # surjections $= m^n - |\cup_{i \in [m]} A_i|$.
- $|\cup_{i \in [m]} A_i| = \sum_{1 \leq i \leq m} |A_i| - \sum_{1 \leq i < j \leq m} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq m} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{m+1} |A_1 \cap \ldots \cap A_m|$
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- What about the summation? terms $1 \leq i < j \leq m = \binom{m}{2}$

Thus, we have # surjections from $[n]$ to $[m]$ =

$m^n - \binom{m}{1}(m-1)^n + \binom{m}{2}(m-2)^n - \ldots + (-1)^{m-1}\binom{m}{m-1} \cdot 1^n$.

Proof of PIE

Theorem: Principle of Inclusion-Exclusion (PIE)

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then,

$$|A_1 \cup \ldots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j|$$
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Proof: (H.W): Prove PIE by induction.

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