CS 105: DIC on Discrete Structures

Instructor : S. Akshay

Oct 05, 2023
Lecture 21 – Counting and Combinatorics
Pigeon-Hole Principle (PHP) and its applications

Principle of Inclusion-Exclusion (PIE)

Theorem: Principle of Inclusion-Exclusion (PIE)

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then,

$$|A_1 \cup \ldots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1} |A_1 \cap \ldots \cap A_n|$$

Proof: (H.W): Prove PIE by induction.

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- What about the summation? terms $1 \leq i < j \leq m = \binom{m}{2}$

Thus, we have # surjections from $[n]$ to $[m]$ =

$m^n - \binom{m}{1}(m-1)^n + \binom{m}{2}(m-2)^n - \ldots + (-1)^{m-1}\binom{m}{m-1} \cdot 1^n$.

Pop Quiz!

1. Does there exist an injective function from a set of $k+1$ elements to a set with k elements? Why or why not?
2. How many cards must be selected from a pack of 52 cards so that at least three cards of the same suit are chosen?
3. Prove or disprove
   3.1 For every $n \in \mathbb{Z}^+$, there exists a multiple of n whose decimal expansion only has $0's$ and $1's$.
   3.2 Every sequence of n^2+1 distinct real numbers contains a subsequence of length $n+1$ which is either increasing or decreasing.
   3.3 If there are $n \geq 1 + r(\ell - 1)$ objects which are colored with r different colors, then there exist ℓ objects all with the same color.

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Pigeon-Hole Principle (PHP) and its applications

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A simple formulation

Let $k \in \mathbb{N}$. If $k + 1$ (or more) objects are to be placed in k boxes, then at least one box will have 2 (or more) objects.

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How do you prove it?

A simple corollary

- Can a function from a set of $k+1$ or more elements to a set with k elements be injective?

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Another simple application

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- Consider $n+1$ integers $k_1 = 1$, $k_2 = 11$, $k_3 = 111$, $\ldots$, $k_{n+1} = 1\ldots1$ (with $n+1$ 1's).

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- When any integer is divided by n, the remainder can be either $0, 1, \ldots, n-1$, i.e., n choices.
- So among the $n+1$ integers, by PHP, at least 2 must have the same remainder.
- That is, $\exists i, j$, $k_i = pn + d$, $k_j = qn + d$.
- But then $|k_i - k_j|$ is a multiple of n and its decimal expansion only has $0's$ and $1's$. $\square$

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- Thus, $N = 9$.

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   $i_k =$ length of longest increasing subsequence starting from a_k
   $d_k =$ length of longest decreasing subsequence starting from a_k
3. Suppose, there are no increasing/decreasing subsequences of length $n + 1$. Then $\forall k$, $i_k \leq n$ and $d_k \leq n$.
4. $\therefore$ by PHP, $\exists \ell, m, 1 \leq \ell < m \leq n^2 + 1$ s.t. $(i_\ell, d_\ell) = (i_m, d_m)$
5. We will show that this is not possible:
   - Case 1: $a_\ell < a_m$. Then $i_m \geq i_\ell + 1$, a contradiction.
   - Case 2: $a_\ell > a_m$. Then $d_\ell \geq d_m + 1$, a contradiction.
6. All a_i's are distinct so this completes the proof. □