CS 105: DIC on Discrete Structures

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Oct 09, 2023

Lecture 22 – Counting and Combinatorics

Pigeon-Hole Principle (PHP) and its extensions

Recap: Topics in Combinatorics

Counting techniques and applications

1. Basic counting techniques, double counting
2. Binomial theorem, permutations and combinations, Estimating $n!$
3. Recurrence relations and generating functions
4. Principle of Inclusion-Exclusion (PIE) and its applications.
   - Hand-shake Lemma
   - Counting the number of surjections on $[n]$.
   - Number of derangements $->\frac{1}{e}$ and more (try them!)

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   - Hand-shake Lemma
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5. Pigeon-Hole Principle (PHP) and its applications.

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Simplest formulation (Variant 0)

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If N objects are placed into k boxes then there is at least one box with at least $\lceil N/k \rceil$ objects.

PHP (Variant 2)

If there are $n \geq 1 + r(\ell - 1)$ objects colored with r different colors, then there exist ℓ objects all with the same color.

Applications of PHP

1. Does there exist an injective function from a set of $k+1$ elements to a set with k elements? Why or why not?

2. How many cards must be selected from a pack of 52 cards so that at least three cards of the same suit are chosen?

3. Prove or disprove

   3.1 For every $n \in \mathbb{Z}^+$, there exists a multiple of n whose decimal expansion only has $0's$ and $1's$.

   3.2 Every sequence of n^2+1 distinct real numbers contains a subsequence of length $n+1$ which is either increasing or decreasing.

   3.3 (H.W.) If there are $n \geq 1 + r(\ell - 1)$ objects which are colored with r different colors, then there exist ℓ objects all with the same color.

This lecture

Pigeon-Hole Principle (PHP) and its extensions

Let's play a game

The coloring game

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- Can you ever have a draw?

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Lemma

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- 2-coloring of edges: coloring all edges of the graph using atmost 2 colors.
- monochromatic (triangle): all edges have the same color.

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- Let $1, \ldots, 6$ be the points, and red/blue the colors.
- Consider the edges $16, 26, 36, 46, 56$.

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- Let $1, \ldots, 6$ be the points, and red/blue the colors.
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- By PHP at least 3 of them must be same color, say $16, 26, 36$ are red.

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Optimality: 6 is the smallest such number

For any graph on 5 or less nodes the above lemma does not hold.

Another coloring problem...

Theorem

Any 2-coloring (say red and blue) of a graph on 10 nodes has either a red triangle or a blue complete graph on 4 nodes.

- complete: all pairs of edges are present.
- How do you prove this? Any ideas?
- How is this different from the previous problem?

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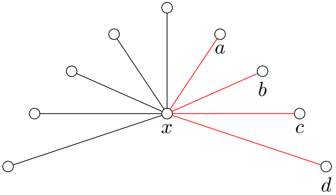

- Consider all edges from some node x.
- Either $\geq$ 4 edges have red color or $\geq$ 6 have blue (why?).

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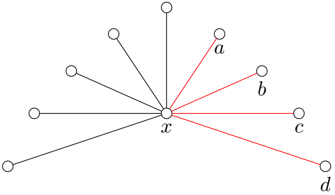

- Case 1: ≥ 4 red edges
  - Either one of edges between a, b, c, d is red or all are blue. So, we are done.

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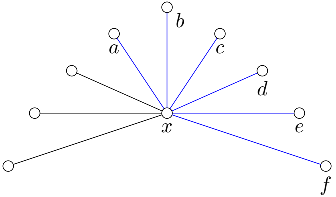

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  - But this means that there are 6 nodes $a, \ldots f$.

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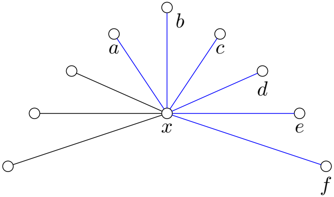

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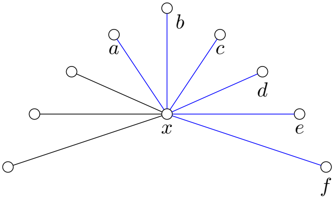

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- And this completes the proof. $\square$

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- But, is this optimal?
- That is, does this fail for a graph on 9 nodes?
- Can you find 2-coloring on a graph of 9 nodes such that the statement above does NOT hold?

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Answer: No! In fact, it does hold on 9 nodes!

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- Recall the Handshake lemma!
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- But is this case possible?
- Recall the Handshake lemma!
  - In any graph, the number of nodes having odd degree is even.
- Thus, this case is impossible and we are done. □