CS 105: DIC on Discrete Structures

Graph theory

Basic terminology, Eulerian walks

Lecture 24

Oct 12 2023

Topic 3: Graph theory

Topics covered in the last three lectures

- Pigeon-Hole Principle and its extensions.
- A glimpse of Ramsey theory

Topic 3: Graph theory

Topics covered in the last three lectures

- Pigeon-Hole Principle and its extensions.
- A glimpse of Ramsey theory
- Ramsey numbers $R(k, \ell)$: Existence of regular sub-structures in large enough general structures.

Topic 3: Graph theory

Topics covered in the last three lectures

- Pigeon-Hole Principle and its extensions.
- A glimpse of Ramsey theory
- Ramsey numbers $R(k, \ell)$: Existence of regular sub-structures in large enough general structures.
- Structures = Graphs,

Topic 3: Graph theory

Topics covered in the last three lectures

- Pigeon-Hole Principle and its extensions.
- A glimpse of Ramsey theory
- Ramsey numbers $R(k, \ell)$: Existence of regular sub-structures in large enough general structures.
- Structures = Graphs, regular= monochromatic/complete

Topic 3: Graph theory

Topics covered in the last three lectures

- Pigeon-Hole Principle and its extensions.
- A glimpse of Ramsey theory
- Ramsey numbers $R(k, \ell)$: Existence of regular sub-structures in large enough general structures.
- Structures = Graphs, regular= monochromatic/complete

Next topic

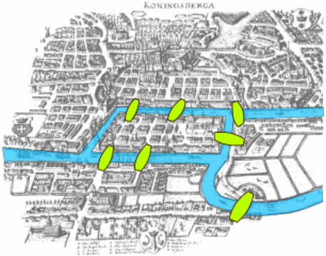
Graphs and their properties!

Topic 3: Graph theory

Textbook Reference

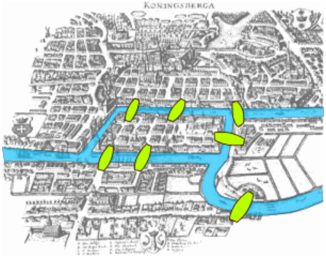
- Introduction to Graph Theory, 2^{nd} Ed., by Douglas West.
- Low cost Indian edition available, published by PHI Learning Private Ltd.

Königsberg Bridge problem



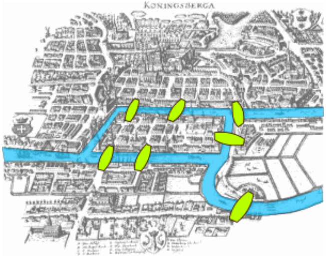
- In 18th century Prussia, the city on river Pregel...
- Find a walk from home, crossing every bridge exactly once and returning home.

Königsberg Bridge problem



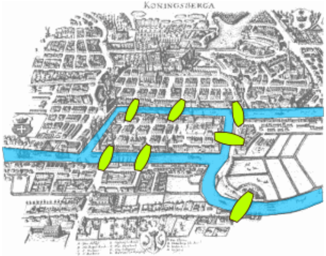
- In 18th century Prussia, the city on river Pregel...
- Find a walk from home, crossing every bridge exactly once and returning home.
- They couldn't find the answer, so in 1735 they asked Leonard Euler, mathematician in St. Petersburg.

Königsberg Bridge problem



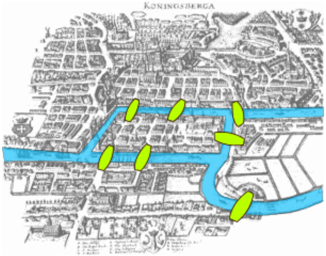
- In 18th century Prussia, the city on river Pregel...
- Find a walk from home, crossing every bridge exactly once and returning home.
- They couldn't find the answer, so in 1735 they asked Leonard Euler, mathematician in St. Petersburg.

Königsberg Bridge problem



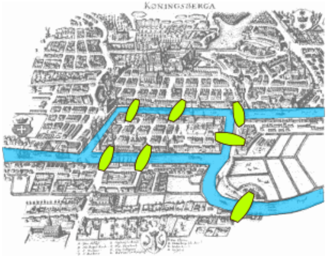
- In 18th century Prussia, the city on river Pregel...
- Find a walk from home, crossing every bridge exactly once and returning home.
- "*This question is so banal, but seemed to me worthy of attention in that [neither] geometry, nor algebra, nor even the art of counting was sufficient to solve it.*"

Königsberg Bridge problem

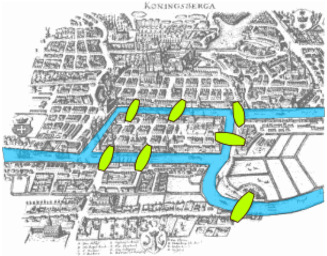


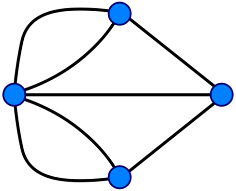
- In 18th century Prussia, the city on river Pregel...
- Find a walk from home, crossing every bridge exactly once and returning home.
- "*This question is so banal, but seemed to me worthy of attention in that [neither] geometry, nor algebra, nor even the art of counting was sufficient to solve it.*"
- Still, he wrote a paper showing that this is impossible!
- Thus, he "gave birth" to the area of graph theory.

Königsberg Bridge problem



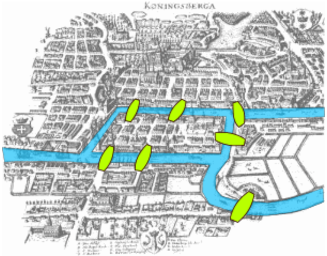
Königsberg Bridge problem

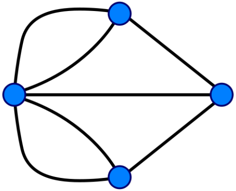




- Qn: Find a walk from home, which crosses every bridge exactly once and returns home.

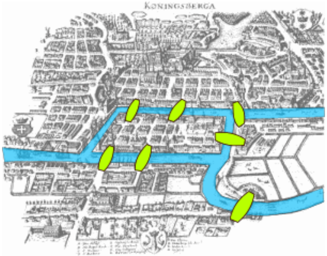
Königsberg Bridge problem

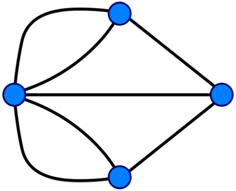




- Qn: Find a walk from home, which crosses every bridge exactly once and returns home.
- Leonard Euler showed in 1735 that this is impossible. The argument is as follows:

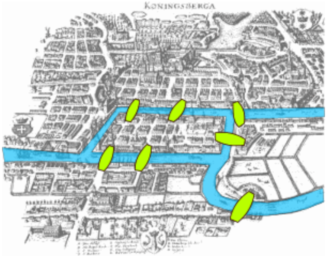
Königsberg Bridge problem

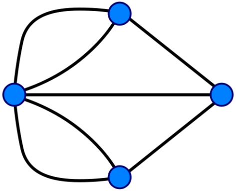




- Qn: Find a walk from home, which crosses every bridge exactly once and returns home.
- Leonard Euler showed in 1735 that this is impossible. The argument is as follows:
  - Each time you enter or leave a vertex, you use an edge.

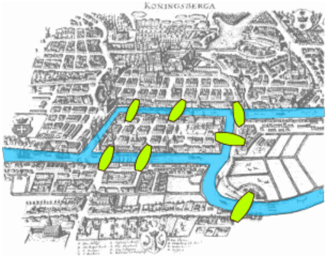
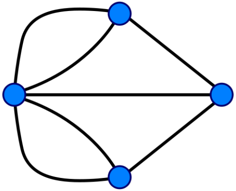
Königsberg Bridge problem





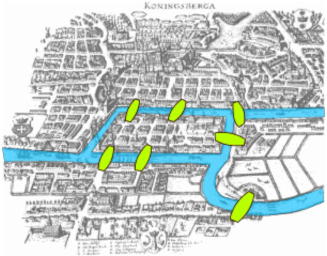
- Qn: Find a walk from home, which crosses every bridge exactly once and returns home.
- Leonard Euler showed in 1735 that this is impossible. The argument is as follows:
  - Each time you enter or leave a vertex, you use an edge.
  - So each vertex must be connected to an even no. of vertices.

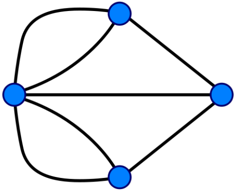
Königsberg Bridge problem

- Qn: Find a walk from home, which crosses every bridge exactly once and returns home.
- Leonard Euler showed in 1735 that this is impossible. The argument is as follows:
  - Each time you enter or leave a vertex, you use an edge.
  - So each vertex must be connected to an even no. of vertices.
  - Which is not the case here, hence it is impossible.

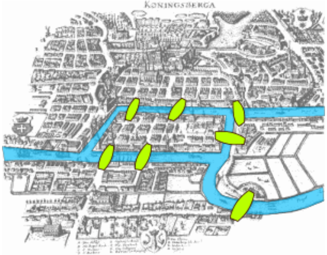
Königsberg Bridge problem

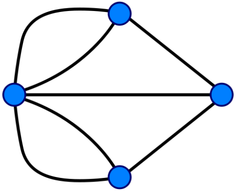




- Qn: Find a walk from home, which crosses every bridge exactly once and returns home.
- Leonard Euler showed in 1735 that this is impossible. The argument is as follows:
  - Each time you enter or leave a vertex, you use an edge.
  - So each vertex must be connected to an even no. of vertices.
  - Which is not the case here, hence it is impossible.
- Clearly, this is a sufficient condition, but is it necessary?

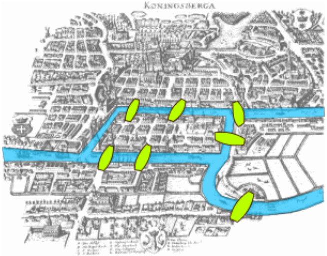
Königsberg Bridge problem

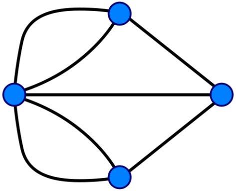




- Qn: Find a walk from home, which crosses every bridge exactly once and returns home.
- Leonard Euler showed in 1735 that this is impossible. The argument is as follows:
  - Each time you enter or leave a vertex, you use an edge.
  - So each vertex must be connected to an even no. of vertices.
  - Which is not the case here, hence it is impossible.
- Clearly, this is a sufficient condition, but is it necessary?
- If every vertex is connected to an even no. of vertices in a graph, is there such a walk?

Königsberg Bridge problem





- Qn: Find a walk from home, which crosses every bridge exactly once and returns home.
- Leonard Euler showed in 1735 that this is impossible. The argument is as follows:
  - Each time you enter or leave a vertex, you use an edge.
  - So each vertex must be connected to an even no. of vertices.
  - Which is not the case here, hence it is impossible.
- Clearly, this is a sufficient condition, but is it necessary?
- If every vertex is connected to an even no. of vertices in a graph, is there such a walk? This is called Eulerian walk.

What are graphs

What are graphs

Definition

A simple graph G is a pair (V, E) of a set of vertices/nodes V and edges E between unordered pairs of vertices called end-points: $e = vu$ means that e is an edge between v and u $(u \neq v)$.

What are graphs

Definition

A simple graph G is a pair (V, E) of a set of vertices/nodes V and edges E between unordered pairs of vertices called end-points: $e = vu$ means that e is an edge between v and u $(u \neq v)$.

- What about loops?
- What about directed edges?
- What about multiple edges?

What are graphs

Definition

A simple graph G is a pair (V, E) of a set of vertices/nodes V and edges E between unordered pairs of vertices called end-points: $e = vu$ means that e is an edge between v and u $(u \neq v)$.

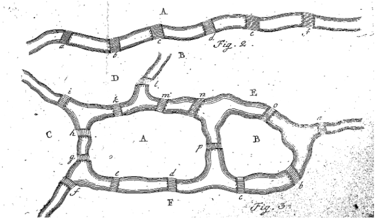
- What about loops?
- What about directed edges?
- What about multiple edges?

General Definition

A graph G is a triple V, E, R where V is a set of vertices, E is a set of edges and $R \subseteq E \times V \times V$ is a relation that associates each edge with two vertices called its end-points.

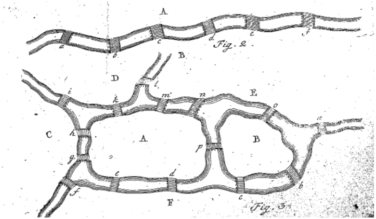
We will consider only finite graphs (i.e., $|V|, |E|$ are finite) and often deal with simple graphs.

Basic terminology



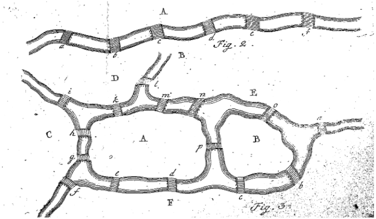
Ex. Draw the graphs!

Basic terminology



The degree $d(v)$ of a vertex v (in an undirected loopless graph) is the number of edges incident to it, i.e., $|\{vw \in E \mid w \in V\}|$.
A vertex of degree 0 is called an isolated vertex.

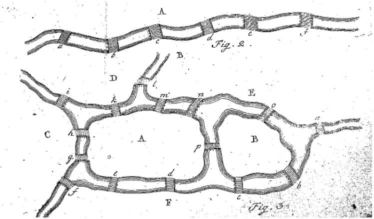
Basic terminology



The degree $d(v)$ of a vertex v (in an undirected loopless graph) is the number of edges incident to it, i.e., $|\{vw \in E \mid w \in V\}|$. A vertex of degree 0 is called an isolated vertex.

- A walk is a sequence of vertices $v_1, \dots v_k$ such that $\forall i \in \{1, \dots k-1\}, (v_i, v_{i+1}) \in E$. The vertices v_1 and v_k are called the end-points and others are called internal vertices.

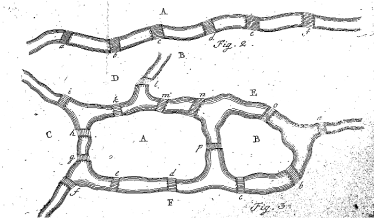
Basic terminology



The degree $d(v)$ of a vertex v (in an undirected loopless graph) is the number of edges incident to it, i.e., $|\{vw \in E \mid w \in V\}|$.
A vertex of degree 0 is called an isolated vertex.

- A walk is a sequence of vertices $v_1, \ldots v_k$ such that $\forall i \in \{1, \ldots k-1\}$, $(v_i, v_{i+1}) \in E$. The vertices v_1 and v_k are called the end-points and others are called internal vertices.
- A walk is called closed if it starts and ends with the same vertex, i.e., its endpoints are the same.

Basic terminology



The degree $d(v)$ of a vertex v (in an undirected loopless graph) is the number of edges incident to it, i.e., $|\{vw \in E \mid w \in V\}|$. A vertex of degree 0 is called an isolated vertex.

- A walk is a sequence of vertices $v_1, \ldots v_k$ such that $\forall i \in \{1, \ldots k-1\}$, $(v_i, v_{i+1}) \in E$. The vertices v_1 and v_k are called the end-points and others are called internal vertices.
- A walk is called closed if it starts and ends with the same vertex, i.e., its endpoints are the same.

Graph is connected if there is a walk between any two vertices.

Eulerian graphs

Definition

A graph is called Eulerian if it has a closed walk that contains all edges, and each edge occurs exactly once. Such a walk is called an Eulerian walk.

Eulerian graphs

Definition

A graph is called Eulerian if it has a closed walk that contains all edges, and each edge occurs exactly once. Such a walk is called an Eulerian walk.

- If G is Eulerian, each vertex must have an even degree.
- This is a necessary condition, but is it sufficient?

Eulerian graphs

Definition

A graph is called Eulerian if it has a closed walk that contains all edges, and each edge occurs exactly once. Such a walk is called an Eulerian walk.

- If G is Eulerian, each vertex must have an even degree.
- This is a necessary condition, but is it sufficient?

Theorem

A graph G with no isolated vertices is Eulerian iff it is connected and all its vertices have even degree.

Eulerian graphs

Definition

A graph is called Eulerian if it has a closed walk that contains all edges, and each edge occurs exactly once. Such a walk is called an Eulerian walk.

- If G is Eulerian, each vertex must have an even degree.
- This is a necessary condition, but is it sufficient?

Theorem

A graph G with no isolated vertices is Eulerian iff it is connected and all its vertices have even degree.

Proof:

Eulerian graphs

Definition

A graph is called Eulerian if it has a closed walk that contains all edges, and each edge occurs exactly once. Such a walk is called an Eulerian walk.

- If G is Eulerian, each vertex must have an even degree.
- This is a necessary condition, but is it sufficient?

Theorem

A graph G with no isolated vertices is Eulerian iff it is connected and all its vertices have even degree.

Proof: ($\implies$)

- Suppose G is Eulerian: every vertex has even degree.
  - each passage through a vertex uses two edges (in and out).
  - at the first vertex first edge is paired with last.

Eulerian graphs

Definition

A graph is called Eulerian if it has a closed walk that contains all edges, and each edge occurs exactly once. Such a walk is called an Eulerian walk.

- If G is Eulerian, each vertex must have an even degree.
- This is a necessary condition, but is it sufficient?

Theorem

A graph G with no isolated vertices is Eulerian iff it is connected and all its vertices have even degree.

Proof: $(\implies)$

- Suppose G is Eulerian: every vertex has even degree.
  - each passage through a vertex uses two edges (in and out).
  - at the first vertex first edge is paired with last.
- Any two edges are in the same walk implies graph is connected (unless it has isolated vertices).