CS 105: DIC on Discrete Structures

Graph theory

Basic terminology, Applications of Eulerian graphs, Bipartite graphs

Lecture 26
Oct 17 2023

Principle of extremality

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- E.g., In the prev proof, since a maximal path could not be extended, we got that every neighbour of an endpoint of a maximal P is in P.
- (H.W) Can you show the theorem directly from extremality without using induction?

A quick quiz

A practical issue

If we want to draw a given connected graph G on paper, how many times must we stop and move the pen? No segment should be drawn twice.

Applications of Eulerian graphs

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Every graph with all vertices having even degree decomposes into cycles

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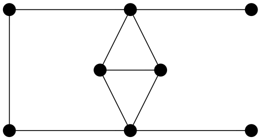

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- Thus, we have shown that at least $\max\{k, 1\}$ trails are required.

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- If $k = 0$, one trail suffices (i.e., an Eulerian walk by previous Thm)
- If $k > 0$ we need to prove that k trails suffice.
  - Pair up odd vertices in G (in any order) and form G' by adding an edge between them.
  - G' is connected, by previous Thm has an Eulerian walk C.
  - Traverse C in G' and for each time we cross an edge of G' not in G, start a new trail (lift pen!).
  - Thus, we get k trails decomposing G. □

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- We have already seen some: connected graphs.
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- Are there other interesting classes of graphs?

Bipartite graphs

Definition

A graph is called bipartite, if the vertices of the graph can be partitioned into $V = X \cup Y$, $X \cap Y = \emptyset$ s.t., $\forall e = (u, v) \in E$,

- either $u \in X$ and $v \in Y$
- or $v \in X$ and $u \in Y$

Example: m jobs and n people, k courses and ℓ students.

- How can we check if a graph is bipartite?
- Can we characterize bipartite graphs?

Characterizing bipartite graphs using cycles.

- Recall: A path or a cycle has length n if the number of edges in it is n.
- A path (or cycle) is call odd (or even) if its length is odd (or even, respectively).

Lemma

Every closed odd walk contains an odd cycle.

Proof: By induction on the length of the given closed odd walk. Exercise!

Characterizing bipartite graphs using cycles.

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Theorem, Konig, 1936

A graph is bipartite iff it has no odd cycle.