CS 105: DIC on Discrete Structures

Graph theory

Graph Isomorphism

Lecture 28
Oct 23 2023

Topic 3: Graph theory

Recap of last four lectures:

1. Basics: graphs, paths, cycles, walks, trails; connected graphs.
2. Eulerian graphs and a characterization in terms of degrees of vertices.
3. Bipartite graphs and a characterization in terms of odd length cycles.

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4. Graph representation and isomorphism

Reference: Sections 1.1-1.3 of Chapter 1 from Douglas West.

Recall: Representing and comparing graphs

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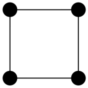

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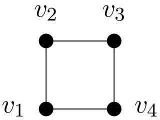

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- As an adjacency list:

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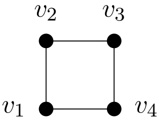

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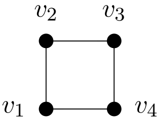

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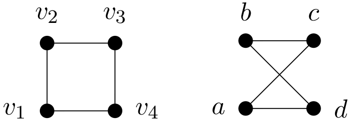

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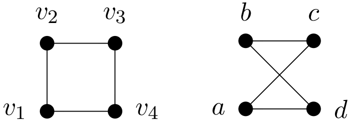

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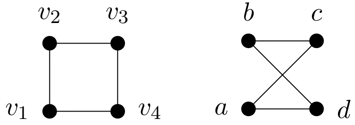

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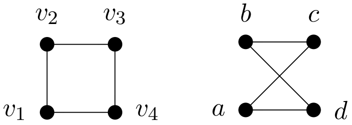

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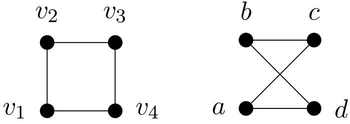

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- How do we formalize this?

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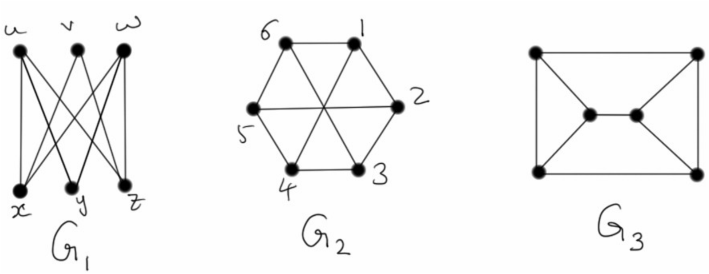
- When we talked about an "unlabeled" graph till now, we actually meant the isomorphism class of that graph!

Graph isomorphism



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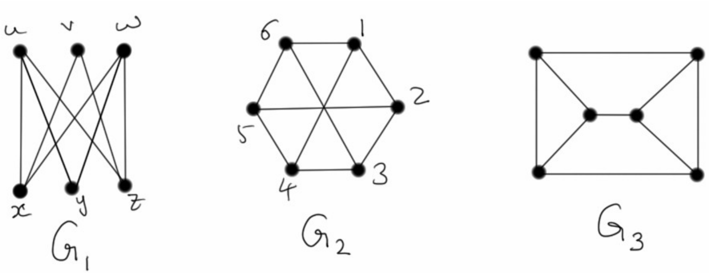

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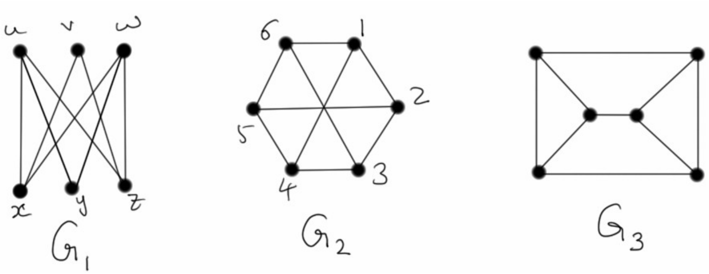

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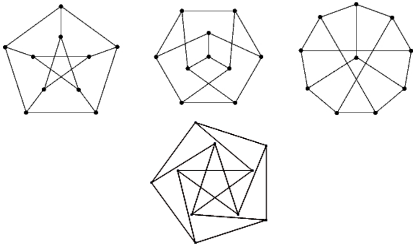
- To show that two graphs are isomorphic, you have to
  1. give names to vertices
  2. specify a bijection
  3. check that it preserves the adjacency relation
- To show that two graphs are non-isomorphic, find a structural property that is different.

Is checking graph isomorphism easy?



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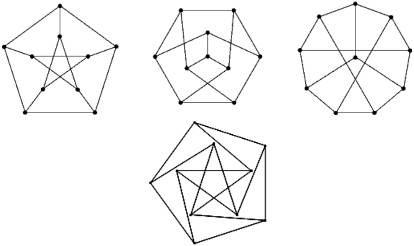

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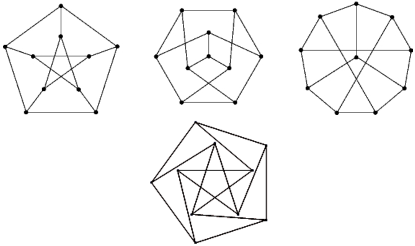

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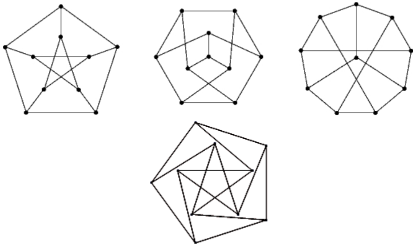

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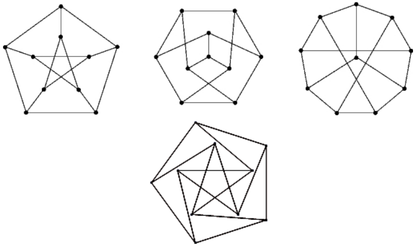

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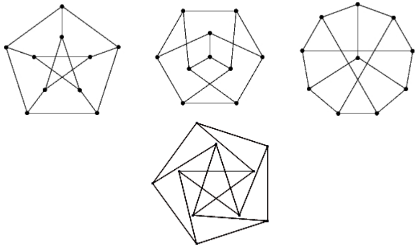

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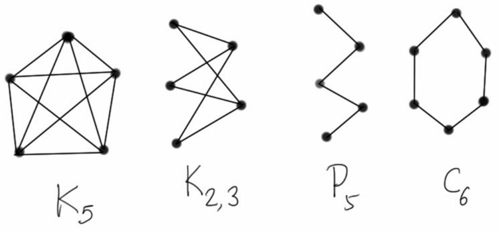
Further reading: Graph and sub-graph isomorphism problems.

Some special graphs and notations



- Complete graphs K_n
- Complete bipartite graphs $K_{i,j}$
- Paths P_n
- Cycles C_n

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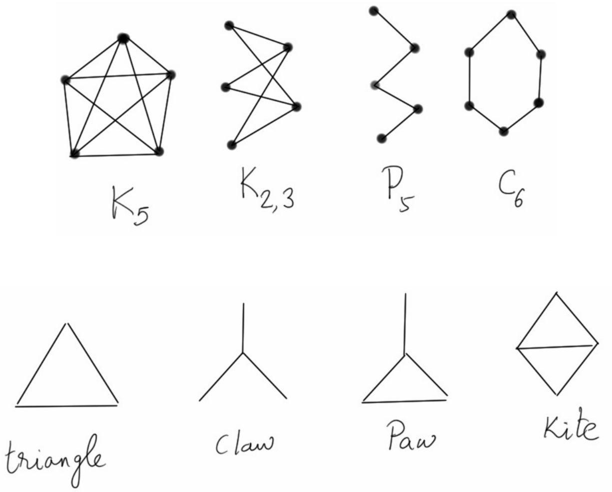

Figure: A whole graph zoo!

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- Are C_5 and $P_5 \cup \{e\}$ isomorphic?

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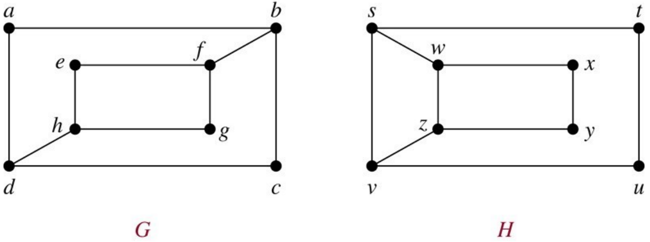

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5. G is bipartite iff H is bipartite.
6. $\dots$

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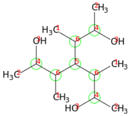
Automorphisms are a measure of symmetry.

Practical applications in graph drawing, visualization, molecular symmetry, structured boolean satisfiability, formal verification



Symmetry Classes
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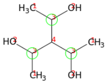
(rule $_2$b')



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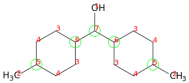

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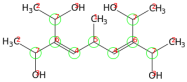

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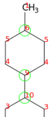
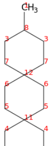
(rule $_3$)





Some basic stuff that we have already seen

Degree-Sum Formula (also called Handshake Lemma!)

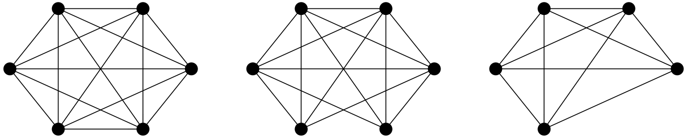
For any graph G with vertex set V and edge set E:

$$\sum_{v \in V} d(v) = 2|E|$$

Some basic stuff that we have already seen

Subgraphs of a graph G

A subgraph H of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ (and the assignment of endpoints to edges in H is same as in G).



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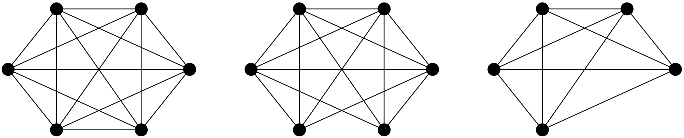

- E.g., a path in a graph G is a subgraph of G.
- A maximal path H is a subgraph of G s.t. there is no other path H' in G such that H is a subgraph of H'.

Some basic stuff that we have already seen

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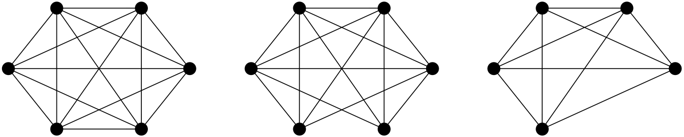
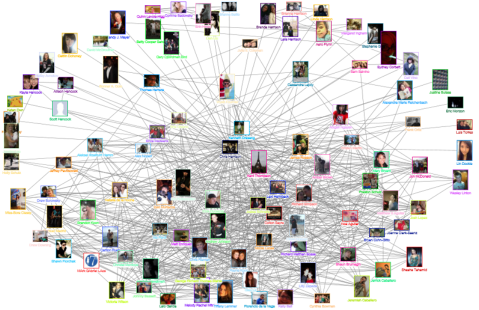


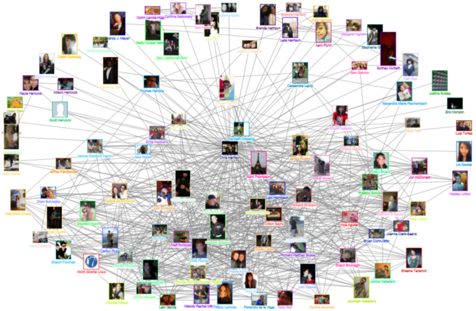
- E.g., a path in a graph G is a subgraph of G.
- A maximal path H is a subgraph of G s.t. there is no other path H' in G such that H is a subgraph of H'.
- Let us now consider some special subgraphs...

Cliques and independent sets



- Consider a large social network graph where friends are linked by an edge.
- What is the largest clique of friends?
- If we want to spread a youtube video, how many people should we send it to so that we are guaranteed everyone will see it (assuming friends forward to each other)?

Cliques and independent sets



Cliques and independent sets

- A clique in a graph is a set of pairwise adjacent vertices.
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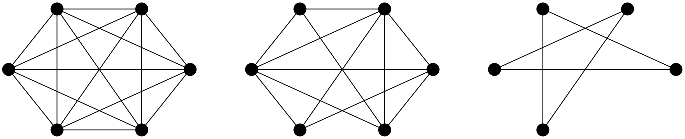
Size of a clique/independent set is the number of vertices in it.



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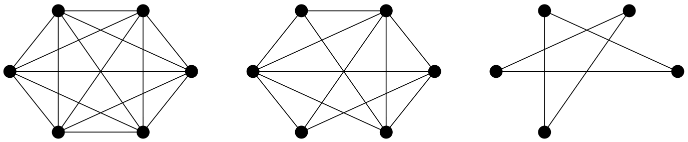

- Thus, a clique in a graph G is a complete subgraph of G.

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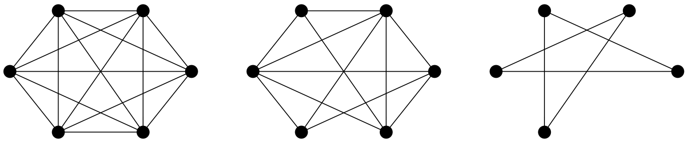

- Thus, a clique in a graph G is a complete subgraph of G.
- An independent set in G is a complete subgraph of $\overline{G}$, where $\overline{G}$ is the complement of G obtained by making all adjacent vertices non-adjacent and vice versa.

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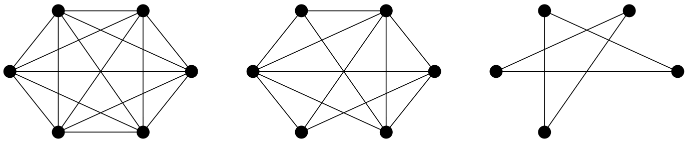

Questions:

- What is the size of the largest clique/independent set in each of the above graphs? In any complete graph?

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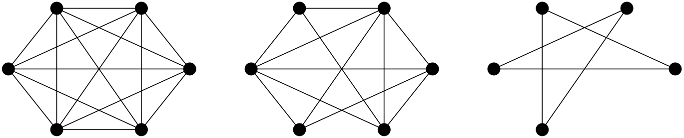

Questions:

- What is the size of the largest clique/independent set in each of the above graphs? In any complete graph?
- Given graph G, integer k, does G have a clique of size k?

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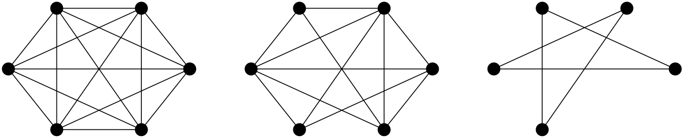

Questions:

- In a graph with 6 vertices, can you always find a clique or an independent set of size 3?

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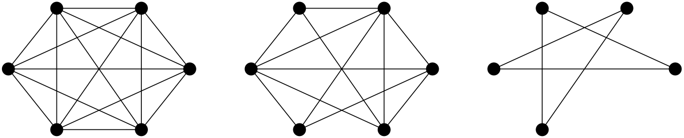

Questions:

- In a graph with 6 vertices, can you always find a clique or an independent set of size 3?
- Yes, because $R(3,3) = 6$!

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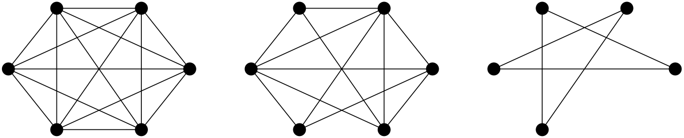

Ramsey's theorem - restated

In any graph with $R(k, \ell)$ vertices, there exists either a clique of size k or an independent set of size ℓ.