# CS 105: DIC on Discrete Structures

## Graph theory

### Connectedness in graphs

Lecture 29
Oct 26 2023

# Topic 3: Graph theory

## Recap

1. Basic definitions: graphs, paths, cycles, walks, trails; connected graphs.
2. Eulerian graphs and a characterization in terms of degrees of vertices.
3. Bipartite graphs and a characterization in terms of odd length cycles.
4. Graph representation (as matrices, lists, etc.)
5. Graph isomorphisms and automorphisms
6. Subgraphs:
    - Cliques and independent sets,
    - A zoo of graphs.

Reference: Sections 1.1-1.3 of Chapter 1 from Douglas West.

# Recall: Graph Automorphisms

**Definition**

An isomorphism from simple graph $G$ to $H$ is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ iff $f(u)f(v) \in E(H)$.

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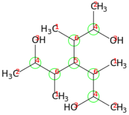
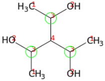
Automorphisms are a measure of symmetry.

Practical applications in graph drawing, visualization, molecular symmetry, structured boolean satisfiability, formal verification



Symmetry Classes
Stereogenic Atoms
(rule $_2$b')



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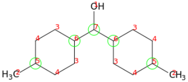

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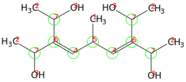

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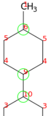
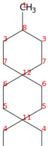
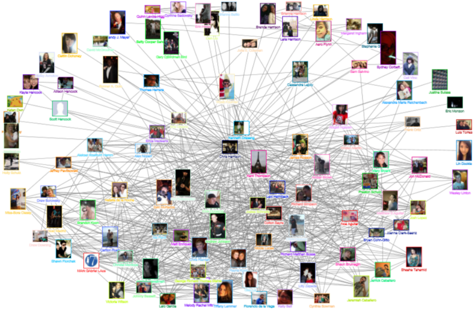


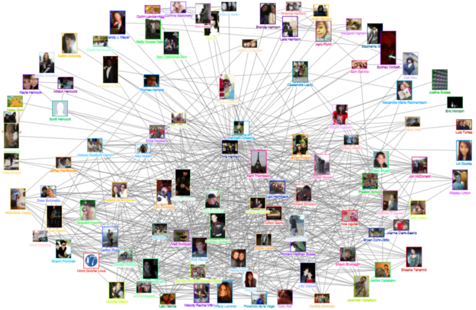


# Cliques and independent sets



- Consider a large social network graph where friends are linked by an edge.
- What is the largest clique of friends?
- If we want to spread a youtube video, how many people should we send it to so that we are guaranteed everyone will see it (assuming friends forward to each other)?

# Cliques and independent sets



## Cliques and independent sets

- A clique in a graph is a set of pairwise adjacent vertices.
- An independent set in a graph is a set of pairwise non-adjacent vertices.

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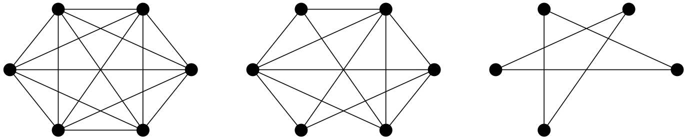

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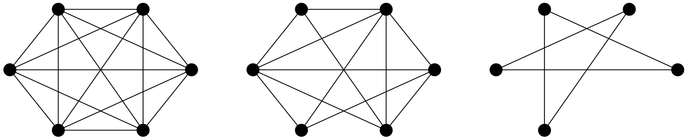

- Thus, a clique in a graph $G$ is a complete subgraph of $G$.

**Subgraphs of a graph $G$**

A subgraph $H$ of a graph $G$ is a graph $H$ such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ (and the assignment of endpoints to edges in $H$ is same as in $G$).

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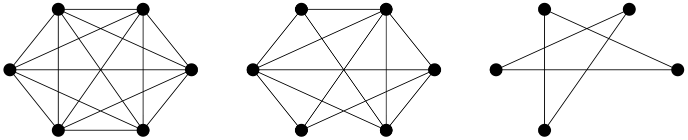

- Thus, a clique in a graph $G$ is a complete subgraph of $G$.
- An independent set in $G$ is a complete subgraph of $\overline{G}$, where $\overline{G}$ is the complement of $G$ obtained by making all adjacent vertices non-adjacent and vice versa.

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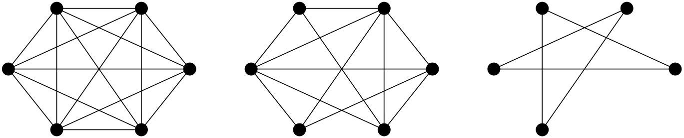

Questions:

- What is the size of the largest clique/independent set in each of the above graphs? In any complete graph?

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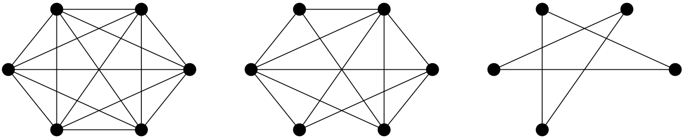

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- Given graph $G$, integer $k$, does $G$ have a clique of size $k$?

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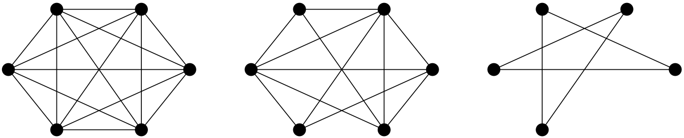

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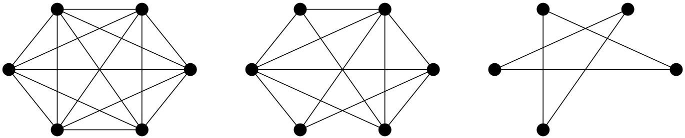

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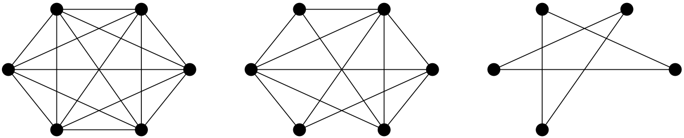

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**Ramsey's theorem - restated**

In any graph with $R(k,\ell)$ vertices, there exists either a clique of size $k$ or an independent set of size $\ell$.

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**Definition**

A (connected) component of $G$ is a maximal connected subgraph, i.e., a subgraph that is connected and is not contained in any other connected subgraph of $G$.

Thus, equivalence classes of $P$ are the vertex sets of the components of $G$.

# Recall: Difference between maximal and maximum

- Is every maximal path maximum, i.e., have maximum length?
- A maximal structure is a structure that is not contained in a larger structure, i.e., increasing the structure will violate some property.
- Maximum just means that size is the greatest among all possible.

**Exercises!**

1. Give a path which is maximal but not maximum.
2. Give a subgraph of a graph which is maximally connected, but not maximum (i.e., does not have maximum # edges).
3. How many maximal/maximum independent sets does $K_{r,s}$ have?

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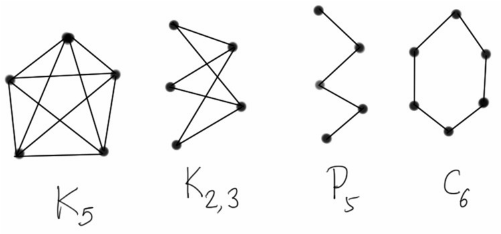

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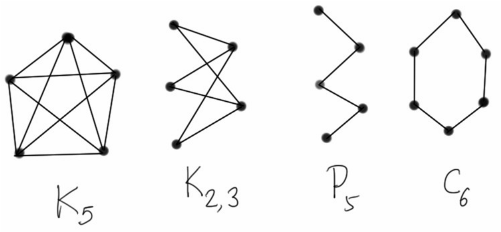

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## Theorem: Characterize cut-edges using cycles

Exercise! An edge is a cut-edge iff it belongs to no cycle.