

CS 105: DIC on Discrete Structures

Graph theory

Hall's theorem and its applications

Lecture 32

Nov 02 2023

Topic 3: Graph theory

Basic definitions and concepts

Characterizations

1. **Eulerian graphs:** Using degrees of vertices.
2. **Bipartite graphs:** Using odd length cycles.
3. **Connected components:** Using cycles.
4. **Maximum matchings:** Using augmenting paths.

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5. **Perfect matchings in bipartite graphs:** Using neighbour sets. – **Hall's theorem**

Recap: Matchings

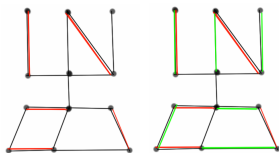
Definitions

- ▶ **Matching:** set of edges with no shared end-points.
- ▶ The vertices incident to edges in a matching are called **saturated**. Others are **unsaturated**.
- ▶ **Perfect matching:** saturates every vertex in graph.
- ▶ **Maximum matching:** matching of maximum size (# edges).
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Theorem

A matching M in G is a maximum matching iff G has no M -augmenting path.

Characterizing perfect matchings in bipartite graphs

Theorem (Hall'35)

A bipartite graph G with bipartitions X, Y has a matching that saturates X iff for all $S \subseteq X$, $|N(S)| \geq |S|$.

- ▶ For $v \in V$, its neighbour-set $N(v) = \{u \in V \mid (u, v) \in E\}$.
- ▶ For $S \subseteq V$, $N(S) = \{u \in V \mid (u, v) \in E \text{ for some } v \in S\}$.

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Proof: (\implies) is straightforward:

- ▶ Let M be a matching.
- ▶ Then for any $S \subseteq X$, each vertex of S is matched to a distinct vertex in $N(S)$
- ▶ So $|N(S)| \geq |S|$.

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- ▶ **Contrapositive:**

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- ▶ If G does not have any matching that saturates X , then surely any maximum matching of G does not saturate X .

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- ▶ If G does not have any matching that saturates X , then surely any maximum matching of G does not saturate X .
- ▶ Let M be such a maximum matching. Then, we will construct $S \subseteq X$ s.t. $|N(S)| < |S|$.

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Proof: (\Leftarrow) Thus, starting from a maximum matching M which does not saturate X , we construct $S \subseteq X$, $|N(S)| < |S|$.

- ▶ Let $u \in X$ be any unsaturated vertex of M .

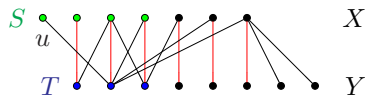
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- ▶ Consider vertices V_u from u by M -alternating paths in G and let $S = V_u \cap X$ and $T = V_u \cap Y$.



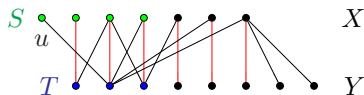
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- ▶ Consider vertices V_u from u by M -alternating paths in G and let $S = V_u \cap X$ and $T = V_u \cap Y$.



Claim: M matches T with $S \setminus \{u\}$ and $|N(S)| = |T|$.

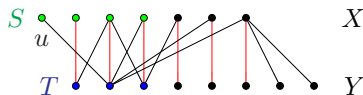
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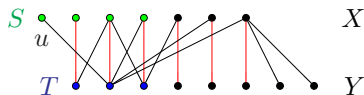
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- ▶ Every vertex of T extends via M to a unique vertex of S .

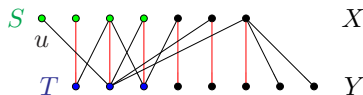
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- ▶ Every vertex of $S \setminus \{u\}$ has an edge in M to a vertex in T .
- ▶ Every vertex of T extends via M to a unique vertex of S .
- ▶ Thus, there is a bijection between T and $S \setminus \{u\}$.

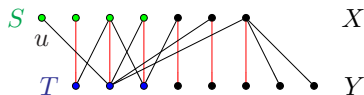
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- ▶ $T \subseteq N(S)$ (from T any M -alternating path will reach S).

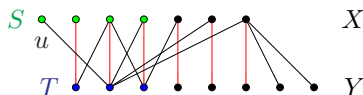
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- ▶ $T \subseteq N(S)$ (from T any M -alternating path will reach S).
- ▶ Conversely, if $v \in S$ has edge to $y \in Y \setminus T$, then path from u to v via M to y is an M -alternating path, implies $y \in T$.

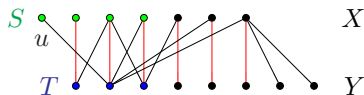
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Thus, $|N(S)| = |T| = |S| - 1 < |S|$

□.

Applications of Hall's condition

Theorem (Hall'35)

A bipartite graph G with bipartitions X, Y has a matching that saturates X iff for all $S \subseteq X$, $|N(S)| \geq |S|$.

The Marriage Theorem (1917)

- ▶ In a group of n women and n men, if every woman is compatible with k men and every man compatible with k women, then a perfect matching must exist!
- ▶ What is the formal statement?

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- ▶ In a group of n women and n men, if every woman is compatible with k men and every man compatible with k women, then a perfect matching must exist!
- ▶ For $k > 0$, every k -regular bipartite graph (i.e, every vertex has degree exactly k) has a perfect matching. Ex. Prove this!

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- ▶ If G is a k -regular X, Y bipartite graph, then $|X| = |Y|$. (why?)

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- ▶ Can you now verify Hall's condition?

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Let's play a game

A two player game on a graph

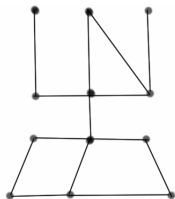
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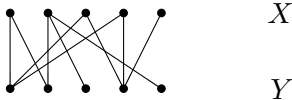


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Theorem (H.W: Qn 3.1.18 from Douglas West)

If G has a perfect matching, then player 2 has a winning strategy; otherwise, player 1 has a winning strategy.