CS 105: DIC on Discrete Structures

Graph theory

Application of Hall's theorem

Lecture 33
Nov 06 2023

Topic 3: Graph theory

Topics in Graph theory

1. Basics concepts and definitions.
2. Eulerian graphs: Using degrees of vertices.
3. Bipartite graphs: Using odd length cycles.
4. Connected components: Using cycles.
5. Maximum matchings: Using augmenting paths.
6. Perfect matchings in bipartite graphs: Using neighbour sets. – Hall's theorem

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6. Perfect matchings in bipartite graphs: Using neighbour sets. – Hall's theorem
7. Today: Applications of Hall's theorem. Matchings in bipartite graphs: Minimum vertex covers. – Konig-Egervary's theorem

Applications of Hall's condition

- Matching: set of edges with no shared end-points.
- The vertices incident to edges in a matching are called saturated. Others are unsaturated.
- Perfect matching: saturates every vertex in graph.

Theorem (Hall'35)

A bipartite graph G with bipartitions X, Y has a matching that saturates X iff for all $S \subseteq X$, $|N(S)| \geq |S|$.

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Application 1: Marriage theorem

- Consider n women and n men. If every woman is compatible with k men and every man compatible with k women, then a perfect matching must exist.
- For $k > 0$, every k-regular bipartite graph (i.e, every vertex has degree exactly k) has a perfect matching.

Application 2

A two player game on a graph

1. Given a graph G, two players will alternatively choose distinct vertices.
2. One player starts by choosing any vertex.
3. Subsequent move must be adjacent to preceding choice (of other player).
4. Thus, we draw a path.
5. Last player who can move wins.

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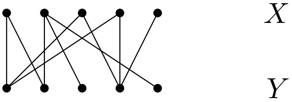

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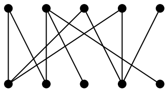
Theorem (Proof: Qn 3.1.18 from Douglas West)

If G has a perfect matching, then player 2 has a winning strategy; otherwise, player 1 has a winning strategy.

Application 3: Another game

Consider the graph of a road network in a city. When a minister is visiting, the Chief of Police wants to place a policeman to watch every road. What is the minimum number of policemen required?



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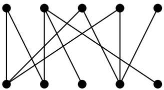

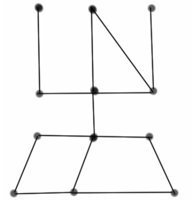

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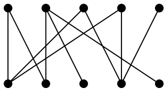

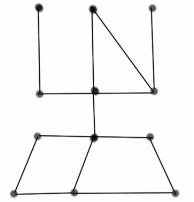

Definition

A vertex cover of a graph G is a set $Q \subseteq V$ that contains at least one endpoint of every edge. Vertices in Q are said to cover E.

Matchings and vertex covers

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1. What is the size of the minimum vertex cover in K_m, $K_{m,n}$?
2. If ℓ is size of maximum matching and k is size of vertex cover,
   2.1 how are ℓ, k related?
   2.2 Give an example of a graph where $k \neq \ell$

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Let's consider bipartite graphs...

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Theorem (Konig '31, Egervary '31)

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- Show that H has a matching that saturates $Q \cap X$ into $Y \setminus T$, H' has a matching saturating T.
- Together this forms the desired matching (since H, H' are disjoint).

□