CS 105: DIC on Discrete Structures

Graph theory

Stable matchings

Lecture 34
Nov 07 2023

Topic 3: Graph theory

Topics in Graph theory

1. Basics concepts and definitions.
2. Eulerian graphs: Using degrees of vertices.
3. Bipartite graphs: Using odd length cycles.
4. Connected components: Using cycles.
5. Maximum matchings: Using augmenting paths.
6. Perfect matchings in bipartite graphs: Using neighbour sets. – Hall's theorem
7. Applications of Hall's theorem: Minimum vertex covers – Konig-Egervary's theorem

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7. Applications of Hall's theorem: Minimum vertex covers – Konig-Egervary's theorem
8. Today: Stable matchings...

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- Together this forms desired matching ($\because$ H, H' are disjoint)

$\square$

Next topic: Stable matchings

Stable matchings

Boys	Girls
• 1	• A
• 2	• B
• 3	• C
• 4	• D
• 5	• E

Stable matchings

	Boys	Girls
$C > B > E > A > D$	• 1	• A : 35214
$ABECD$	• 2	• B : 52143
$DCBAE$	• 3	• C : 43512
$ACDBE$	• 4	• D : 12345
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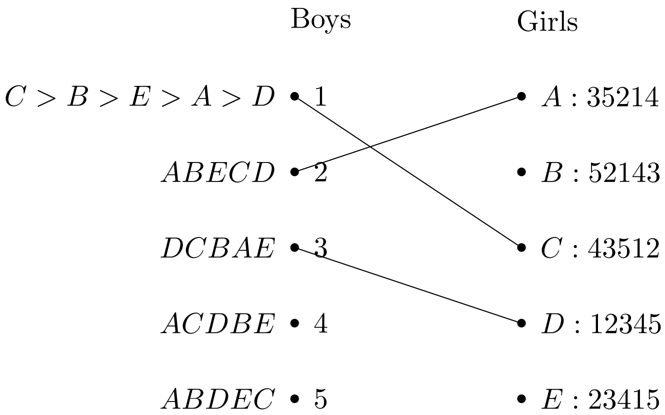
Stable matchings



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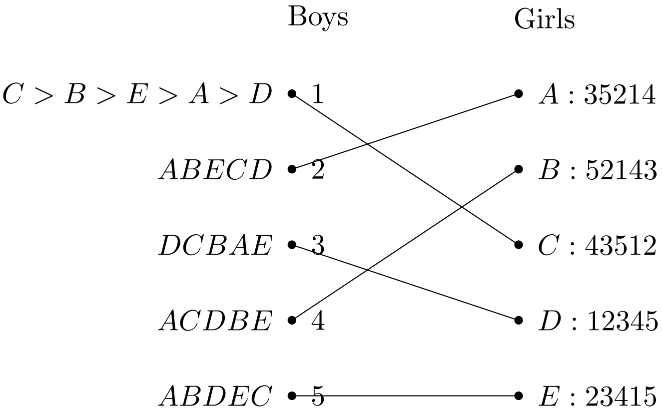

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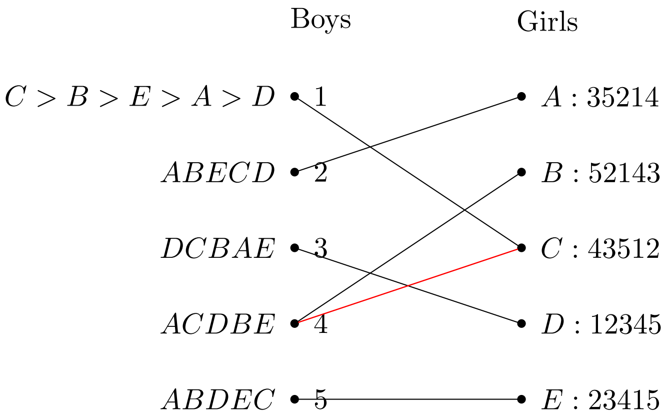

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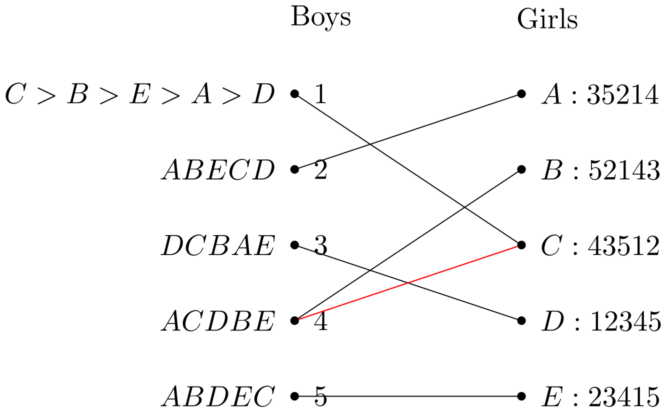

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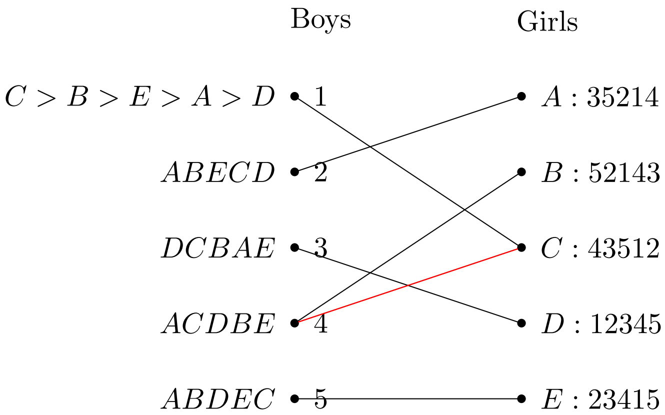

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- Danger! 4 prefers C to B and C prefers 4 to 1. Divorce!
- Qn: Can you match everyone without such Rogue couples?!

More than just a funny puzzle

- College admissions: Original Gale and Shapley paper, 1962.
- Matching hospitals and residents.
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- Matching students with jobs.
- Matching (PG) TAs with courses.
- JEE algorithm...

Stable matchings

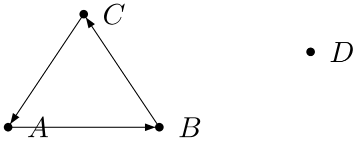
Definition

Given a matching M in a graph with preference lists of nodes.

- Unstable pair: Two vertices x, y such that x prefers y to its assigned vertex and vice versa.
- x, y would be happier by eloping.
- Qn: Find a perfect matching with no unstable pairs. Such a matching is called a Stable Matching.

Roommates Problem

- $A : BCD$
- $B : CAD$
- $C : ABD$
- $D : ABC$



► What can you observe from this?

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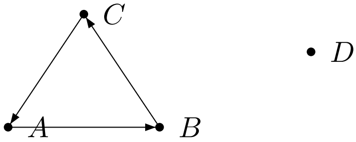

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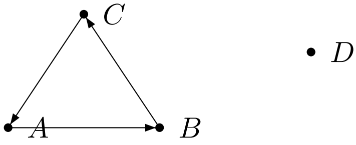

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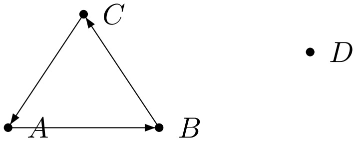

- What can you observe from this?
- Stable matchings don't always exist.
- So, do they exist for bipartite graphs and how can we prove this?

The proposal algorithm

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- Does this algorithm terminate?
- If yes, does it produce a stable matching when it terminates?

Termination and Correctness of the proposal algo

- Try out the algo on the example.