CS 105: DIC on Discrete Structures

Graph theory

Stable matchings, and the end.

Lecture 35
Nov 09 2023

Topic 3: Graph theory

Topics in Graph theory

1. Basics concepts and definitions.
2. Eulerian graphs: Using degrees of vertices.
3. Bipartite graphs: Using odd length cycles.
4. Connected components: Using cycles.
5. Maximum matchings: Using augmenting paths.
6. Perfect matchings in bipartite graphs: Using neighbour sets. – Hall's theorem
7. Applications of Hall's theorem: Minimum vertex covers – Konig-Egervary's theorem
8. Stable matchings...

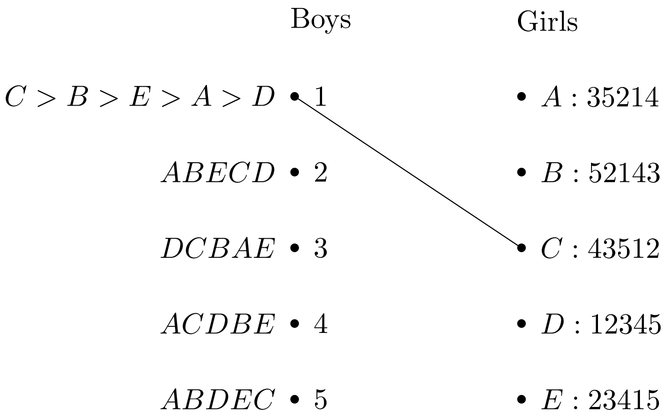
Stable matchings

Boys	Girls
• 1	• A
• 2	• B
• 3	• C
• 4	• D
• 5	• E

Stable matchings

	Boys	Girls
$C > B > E > A > D$	• 1	• A : 35214
$ABECD$	• 2	• B : 52143
$DCBAE$	• 3	• C : 43512
$ACDBE$	• 4	• D : 12345
$ABDEC$	• 5	• E : 23415

Stable matchings



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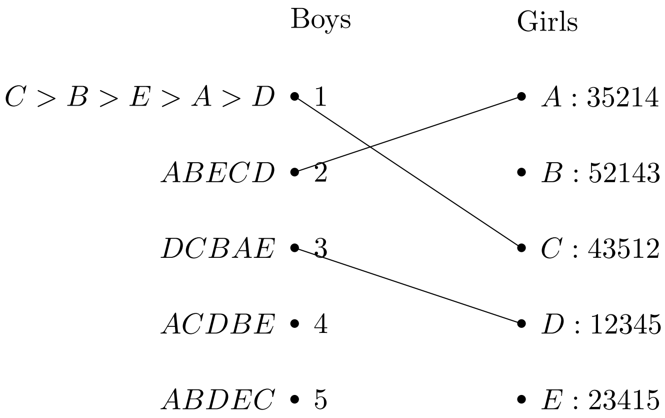
Stable matchings



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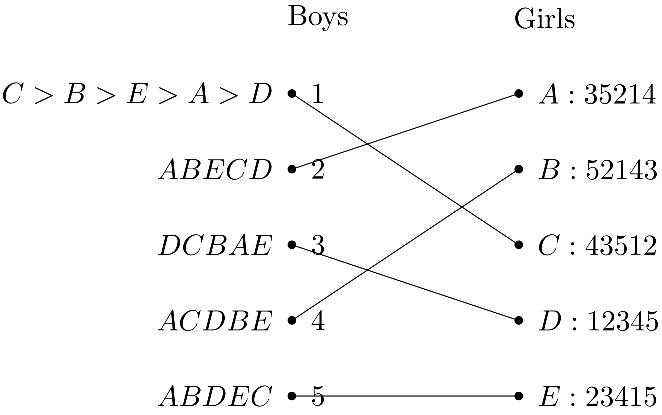

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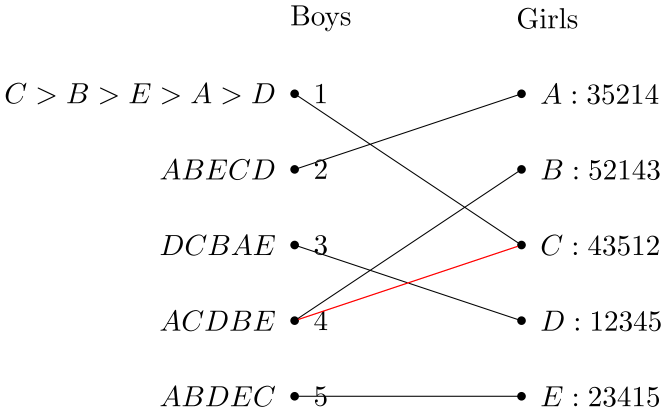

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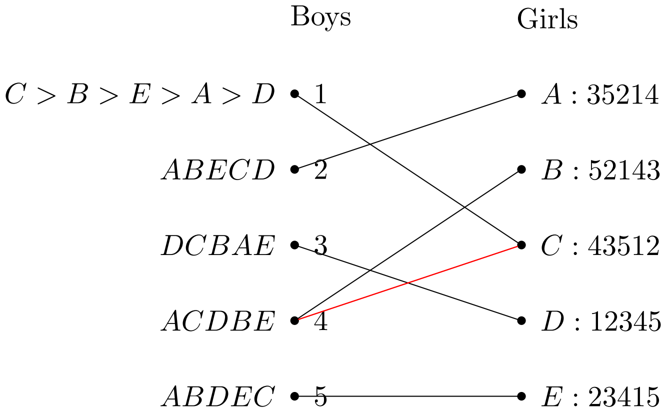

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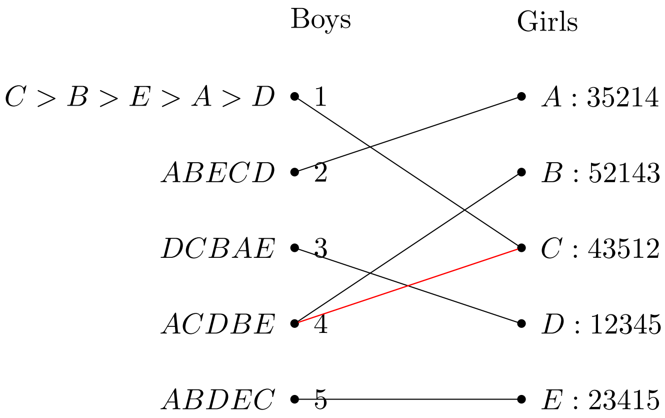

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- Qn: Can you match everyone without such Rogue couples?!

More than just a funny puzzle

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- Matching hospitals and residents.
- Matching dancing partners.
- Matching students with jobs.

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- Matching hospitals and residents.
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- Matching students with jobs.
- Matching (PG) TAs with courses.
- JEE algorithm...

Stable matchings

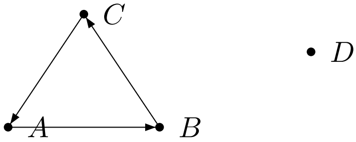
Definition

Given a matching M in a graph with preference lists of nodes.

- Unstable pair: Two vertices x, y such that x prefers y to its assigned vertex and vice versa.
- x, y would be happier by eloping.
- Qn: Find a perfect matching with no unstable pairs. Such a matching is called a Stable Matching.

Roommates Problem

- $A : BCD$
- $B : CAD$
- $C : ABD$
- $D : ABC$



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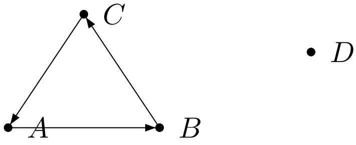

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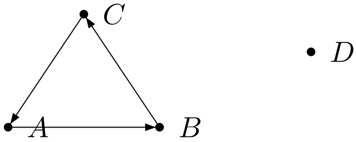

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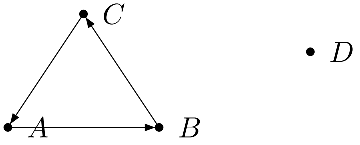

- What can you observe from this?
- Stable matchings don't always exist.
- So, do they exist for bipartite graphs and how can we prove this?

The proposal algorithm

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- Does this algorithm terminate?
- If yes, does it produce a stable matching when it terminates?

Termination and Correctness of the proposal algo

- Try out the algo on the example.

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Lemmas

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    - The algo terminates within n^2 days.
    - For each day (except last), at least one woman is crossed off some man's list.
    - As there are n men and each has list of size n, algo must terminate in n^2 days.

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  - By Lemma 2, she likes her final partner at least as much as M'', so better than M.

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- If (M, W) is pair in current matching, s.t., M prefers W'.
- We will show that W' prefers some other M' and hence no unstable pair.
- Thus no man can be part of an unstable pair, implies stable matching. □

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The male-proposal algorithm is male-optimal (and "woman-pessimal").

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Conclusion: Propose first!

Further reading

- Many questions, rich theory.
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- Can you do better by lying? Boys - no!, Girls - yes!
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- D. Gale and L.S. Shapley, College Admissions and the Stability of Marriage, American Mathematical Monthly 69(1962), pp. 9-14.
- D. Gusfield and R.W. Irving, The Stable Marriage Problem: Structure and Algorithms, MIT Press, 1989.

The 2012 Nobel prize in Economics to Shapley and Roth: "for the theory of stable allocations and the practice of market design".

Summary

What we covered in this course

1. Mathematical proofs and basic structures
2. Counting and combinatorics
3. Introduction to graph theory

Summary: Part 1

- Mathematical proofs and reasoning
- Basic discrete structures

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  - How to reason and write proofs formally
  - Propositions, predicates
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  - Applications: showing impossibility theorems for CS, parallel task scheduling algorithms.

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