CS 105: Department Introductory Course on Discrete Structures

Instructor : S. Akshay

Jul 30, 2024

Lecture 02 – Propositions, Predicates and Theorems

Logistics and recap

Course material, references are being posted at

- http://www.cse.iitb.ac.in/~akshayss/teaching.html

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Recap of last lecture

- What are discrete structures?
- Course outline
- Chapter 1: Proofs and structures
  - Propositions: statements that can be assigned a truth value
  - We denote them using variables $p, q, \ldots$

Propositional calculus and Boolean algebra

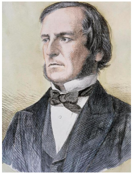

Figure: George Boole (1815 – 1864)

Combining propositions

- $\neg p$: It is not raining

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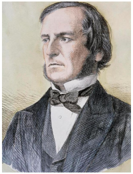

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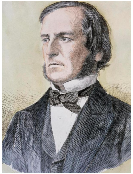

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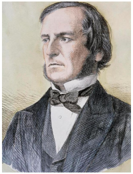

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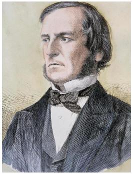

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- If it is raining or there is a sprinkler overhead and I dont have an umbrella, then I will get wet

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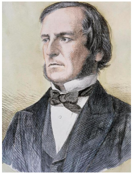

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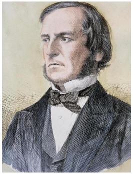

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- $p \leftrightarrow q$: p if and only if q (also written p iff q)
  "same as" or logically equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$

Truth Tables and Logical Equivalence

p	$\neg p$
T	F
F	T

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Logical Equivalence: Truth tables are identical!

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Warning: English can be imprecise, but logic is precise!

Predicates and quantifiers

Moving on...

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Consider again...

$$(n+1)(n-1) = (n^2 - 1)$$
$$x = y + 8$$

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- $\forall n \qquad (n+1)(n-1) = (n^2-1)$
- $\forall x, \exists y, \qquad\qquad x = y + 8$

- $\forall n$ stands for all values of n in a given domain
- $\exists n$ stands for exists n

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Consider again...

- $\forall n \in \mathbb{N}\ (n+1)(n-1) = (n^2 - 1)$
- $\forall x, \exists y, x, y \in \mathbb{Z}\ x = y + 8$

- $\forall n$ stands for all values of n in a given domain
- $\exists n$ stands for exists n
- $\in$ is the element of symbol
- $\mathbb{N}$ stands for all natural numbers
- $\mathbb{Z}$ stands for all integers
- $\mathbb{R}$, $\mathbb{Q}$, ...

Some propositions are not so easy to "determine"...
– e.g., $2^{67} - 1$ is not a prime.

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  - $\forall x(L(x) \vee \exists y(L(y) \wedge F(x, y)))$

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  - $\forall x(L(x) \vee \exists y(L(y) \wedge F(x, y)))$ Fix Domain = students in this class; $L(x) = x$ has a laptop; $F(x, y) = x, y$ are friends.

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    - All students in this class have a laptop or have a friend who has a laptop.

Theorems and proofs

A theorem is a proposition which can be shown true

Classwork: Prove the following theorems.

1. $\neg(p \wedge q)$ is logically equivalent to $\neg p \vee \neg q$
2. For all $a, b, c \in \mathbb{R}^{\geq 0}$, if $a^2 + b^2 = c^2$, then $a + b \geq c$.
3. If 6 is prime, then $6^2 = 30$.
4. For all $x \in \mathbb{Z}$, x is an even iff $x + x^2 - x^3$ is even.

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4. For all $x \in \mathbb{Z}$, x is an even iff $x + x^2 - x^3$ is even.

5. There are infinitely many prime numbers.
6. There exist irrational numbers x, y such that x^y is rational.
7. For all $n \in \mathbb{N}$, $n! \leq n^n$.
8. There does not exist a program which will always determine whether an arbitrary (input-free) program will halt.

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Contrapositive and converse

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- To show A iff B, you have to show A implies B and conversely, B implies A.
- Note the difference between contrapositive and converse.