CS 105: Department Introductory Course on Discrete Structures

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Lecture 05 – Induction and Well-Ordering Principle

Recap

Last week

- What are discrete structures?
- Course outline
- Chapter 1: Proofs and structures
  - Propositions: statements that can be assigned a truth value
  - Predicates: propositions with variables
  - Quantifiers
  - Theorems and different types of proofs

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This week

Induction

Interesting fallacy in using induction!

Conjecture: All horses have the same colour.

"Proof" by induction on number of horses:

1. Base Case ($n = 1$) The case with one horse is trivial.
2. Induction Hypothesis Assume for $n = k \geq 1$, i.e., any set of $k(\geq 1)$ horses has same color.
3. Induction Step We want to show any set of $k + 1$ horses have same color. Consider such a set, say $1, \ldots, k + 1$.
   (A) First, consider horses $1, \ldots, k$. By induction hypothesis, they have same color.
   (B) Next, consider horses $2, \ldots, k + 1$. By induction hypothesis, they have same color.
   (C) Therefore, 1 has same color as 2 (by A) and 2 has same color as $k + 1$ (by B), implies all $k + 1$ have same color.
4. Thus, by induction, we conclude that for all $n \geq 1$, any set of n horses has the same color. □

Where is the bug?

Proof of algorithm using induction

Consider the following algorithm:

input: non-zero real number a, non-negative integer n.

procedure: if $n = 0$, then return $f(a, n) = 1$;

else $f(a, n) = a \cdot f(a, n - 1)$;

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Theorem: Prove that the algorithm computes the function $f(a, n) = a^n$ for all non-negative integers n, $a \in \mathbb{R}^{\neq 0}$.

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- Now for $n = k + 1$, $f(a, k + 1) = a \cdot f(a, k) = a \cdot a^k = a^{k+1}$ (by Induction Hyp).

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- Now for $n = k + 1$, $f(a, k+1) = a \cdot f(a, k) = a \cdot a^k = a^{k+1}$ (by Induction Hyp).
- Thus, by induction for all non-negative integers n, the algorithm above computes $f(a, n) = a^n$. □

What is the basis for induction

Axiom: Induction

Let $P(n)$ be a property of non-negative integers. If

- $P(0)$ is true (Base case)
- for all $k \geq 0$, $P(k) \implies P(k+1)$ (Induction step)

then $P(n)$ is true for all $n \in \mathbb{N}$.

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Every nonempty set of non-negative integers has a smallest element. Does this seem familiar? Obvious? What about for rationals?!

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What about it's converse?

WOP implies induction

Theorem: Well-ordering principle implies Induction

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Proof by contradiction:

1. Suppose Induction is not true. This means that,
   - 1.1 Base Case holds: $P(0)$ is true;
   - 1.2 Induction Step holds: for $\forall n \geq 0$, $P(n) \implies P(n+1)$;
   - 1.3 But the conclusion doesn't hold, i.e., it isn't the case that ($P(n)$ is true for all non-negative integers).

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2. Point (1.3) implies there exists $n \in \mathbb{N}$ s.t., $P(n)$ is false.
3. Now, consider set $S = \{i \in \mathbb{N} \mid P(i) \text{ is false } \}$.

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4. S is a non-empty (due to 2.) set of non-negative integers, hence by WOP, it has a smallest element, say $i_0 \in S$.
5. $i_0 \neq 0$ (due to 1.1) and $i_0 - 1 \notin S$ (since i_0 is smallest in S).
6. $i_0 - 1 \notin S$ implies $P(i_0 - 1)$ is true (by definition of S).
7. By (1.2), $P(i_0)$ must be true, $i_0 \notin S$.Contradiction! $\square$

The Well Ordering Principle and Induction

Well Ordering Principle

Every nonempty set of non-negative integers has a smallest element.

Induction

Let $P(n)$ be a property of non-negative integers. If

- $P(0)$ is true (Base case)
- for all $k \geq 0$, $P(k) \implies P(k+1)$ (Induction step) then $P(n)$ is true for all $n \in \mathbb{N}$.

Theorem: Well-ordering principle iff Induction

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Theorem: Well-ordering principle iff Induction

So, we could have chosen either one of them as our basic axiom!

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- Call this least number n. First, n can't be a prime (why?).
- So $n = a \cdot b$, where $n > a, b > 1$.

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- Since a and b are smaller than the smallest number in S, they can be written as product of one or more primes.
- Let $a = p_1 \dots p_k$ and $b = q_1 \cdots q_l$ for $k, l \geq 1$. But then $n = p_1 \cdots p_k \cdot q_1 \cdots q_l$, which is a contradiction. □

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Qn: How do you show uniqueness? (H.W.)