CS 105: Department Introductory Course on Discrete Structures

Instructor : S. Akshay

Aug 12, 2024

Lecture 07 – Basic structures: sets

Recap of last 6 lectures

Week 01

- What are discrete structures?
- Chapter 1: Proofs and mathematical reasoning
  - Propositions: statements that can be assigned a truth value
  - Predicates and Quantifiers
  - Theorems and types of proofs: contradiction, contrapositive

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Week 02

- Induction
- Well-ordering principle
- Strong Induction

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Two problem-sheets released on Piazza

1. Questions on Basic proofs, reasoning
2. Questions on Induction, WOP, Strong Induction.

Logistics

Optional Problem Solving Sessions started...

- Second to be scheduled soon...

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Pop Quiz Anytime!

Quiz 1: End of August, tentatively Aug 28th, 8.30am

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   lhs$= 1 + 2 + \ldots + k + (k + 1) = (1 + 2 + \ldots k) + (k + 1)$
   $= \frac{k(k+1)}{2} + (k + 1)$ (By Induction Hypothesis)
   $= \frac{k(k+1)+2(k+1)}{2} = \frac{(k+2)(k+1)}{2} = rhs.$

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4. Hence by induction we can conclude for all $n \in \mathbb{N}, n \geq 1$.

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9. Implies $k' \notin S$. A contradiction.

From proofs to structures

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Next: Chapter 2: Basic Discrete Structures

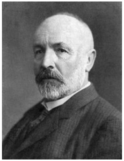
- Finite and Infinite Sets,
- Functions
- Relations

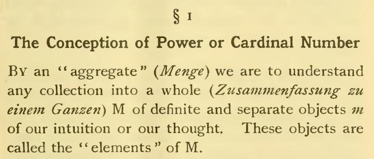
Sets

What is a set?

- A set is an unordered collection of objects.
- The objects in a set are called its elements.

Sets





§ I

The Conception of Power or Cardinal Number

BY an "aggregate" (*Menge*) we are to understand any collection into a whole (*Zusammenfassung zu einem Ganzen*) M of definite and separate objects *m* of our intuition or our thought. These objects are called the "elements" of M.

Figure: Georg Cantor (1845-1918); extract

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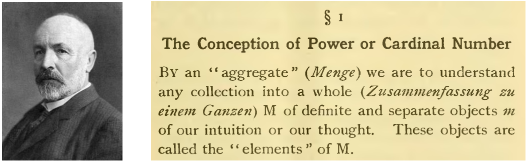

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More formally,

Let P be a property. Any collection of objects that are defined by (or satisfy) P is a set, i.e., $S = \{x \mid P(x)\}$.

Some simple boring stuff about sets

Examples and properties

- We have already seen examples: $\mathbb{Z}, \mathbb{N}, \mathbb{R}$, set of all horses,...
- Let A, B be two sets. Recall the usual definitions:
  - Equality $A = B$, Subset $A \subseteq B$,

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So, what is the difference between $\{\emptyset\}$ and $\emptyset$?

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Barber's paradox: Does the barber shave himself?

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How do you resolve this?

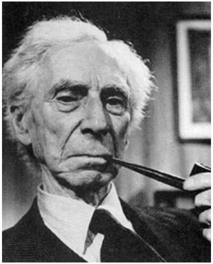

Figure: Bertrand Russell (1872-1970)

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Start with a few objects defined. Then for a set A and a property P, $S = \{x \in A \mid P(x)\}$ is a set.

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Why does this definition get rid of Russell's paradox?

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- if $(S \in S)$: from the definition of S, $S \in A$ and $S \notin S$, which is a contradiction.
- if $(S \notin S)$: from the definition, either $S \notin A$ or $S \in S$. But we have assumed that $S \notin S$. Hence, $S \notin A$. No contradiction!

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- Example: $\{$set of all even natural numbers$\}$ vs $\mathbb{N}$ vs $\mathbb{Q}$ vs $\mathbb{R}$
- Turns out we need functions... but first...

Hilbert's hotel

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3. What if infinitely many more trains with infinitely many more guests arrive? (H.W)