CS 105: Department Introductory Course on Discrete Structures

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Lecture 08 – Basic structures: sets and functions

From proofs to structures

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Chapter 2: Basic Discrete Structures

- Finite and Infinite Sets,
- Functions
- Relations

Hilbert's hotel

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2. What if infinitely many more guests arrive?
3. What if infinitely many more trains with infinitely many more guests arrive? (H.W)

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i.e., $f : A \to B$ is a subset R of $A \times B$ such that

(i) $\forall a \in A, \exists b \in B$ such that $(a, b) \in R$, and

(ii) if $(a, b) \in R$ and $(a, c) \in R$, then $b = c$.

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- We write $f(a) = b$ and call b the image of a.
- $Range(f) = \{b \in B \mid \exists a \in A \text{ s.t. } f(a) = b\}$, $Domain(f) = A$

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Composition of functions

- If $g : A \to B$ and $f : B \to C$, then $f \circ g : A \to C$ is defined by $f \circ g(x) = f(g(x))$.
- Defined only if $Range(g) \subseteq Domain(f)$.
- Exercise: if $f(x) = x^2$, $g(x) = x - x^3$ with $f, g : \mathbb{R} \to \mathbb{R}$, what is $f \circ g(x), g \circ f(x)$?

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Composition of functions is associative

- If $h : A \to B$ and $g : B \to C$ and $f : C \to D$, then $f \circ (g \circ h) = (f \circ g) \circ h$.

Check it! (H.W.)

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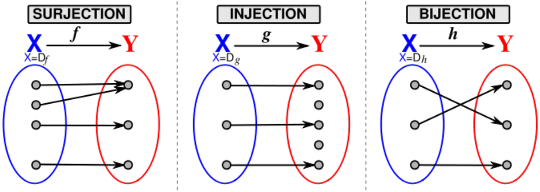
Inverse of a function

- If $f : A \to B$ is a ??? function, then $f^{-1} : B \to A$ defined by $f^{-1}(b) = a$ if $f(a) = b$, is called its inverse.

Comparing (finite and infinite) sets



- Surjective or onto: $f : A \to B$ is surjective if $\forall y \in B$, $\exists x \in A$ such that $f(x) = y$.
- Injective or 1-1: $f : A \to B$ is injective if $\forall x, y \in A$, if $f(x) = f(y)$, then $x = y$.
- Bijective or 1-1 correspondence: A function is bijective if it is surjective and injective.

If f is a bijection, then its inverse function exists and $f \circ f^{-1} = f^{-1} \circ f = \text{id}$

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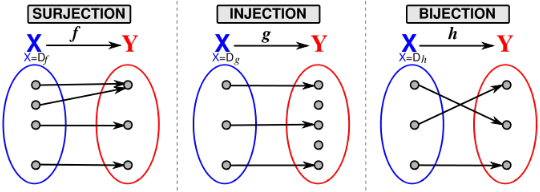

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1. $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x) = x^2$.
2. $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ such that $f(x) = x^2$.

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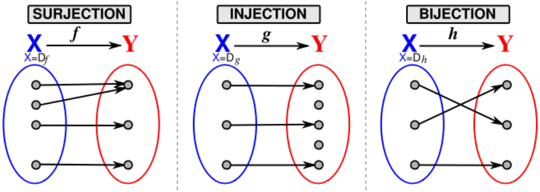

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Properties of finite and infinite sets

Relative notion of “size”

Thus, two finite/infinite sets have the same “size” iff there is a bijection between them.

Properties of finite and infinite sets

Similarities between finite and infinite sets

- $\exists$ **bij** from A to B and B to C, implies $\exists$ **bij** from A to C.
- $\exists$ **bij** from A to B, then $\exists$ **bij** from B to A.
- $\exists$ **surj** from A to B and $\exists$ **surj** B to A, implies $\exists$ **bij** from A to B.

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Differences between finite and infinite sets

- For finite sets, if A is a set and $b \notin A$, then $|A| \neq |A \cup \{b\}|$.
- What about infinite sets?

Difference between finite and infinite sets

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1. Show contrapositive! If A is finite, then there can't be a bijection from A to $A \cup \{b\}$.

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4. Collecting all such $a_i's$, we get a subset $A' = \{a_i \in A \mid i \in \mathbb{N}\} \subseteq A$. (Note it may be that $A \neq A'$).
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- Even if A, B are infinite, $A \subset B$, there can be a bijection from A to B

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- Even if A, B are infinite, $A \subset B$, there can be a bijection from A to B, i.e., they have the same "cardinality".
- From any set A, there is a surjection from A to $\mathbb{N}$.