CS 105: Department Introductory Course on Discrete Structures

Instructor : S. Akshay

Aug 19, 2024

Lecture 09 – Basic structures: Countable sets and functions

Recap of last 8 lectures

Week 01 & 02

- What are discrete structures?
- Chapter 1: Proofs and mathematical reasoning
  - Propositions: statements that can be assigned a truth value
  - Predicates and Quantifiers
  - Theorems and types of proofs: contradiction, contrapositive
  - Induction, WOP, Strong Induction

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Week 03

- Chapter 2: Basic Mathematical Structures
- Sets, functions, infinite sets

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- Chapter 2: Basic Mathematical Structures
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Three problem-sheets released on Piazza

- TA Problem solving session at 7pm
- Venue: Computing Complex 1st floor: CC 105/103

Properties of finite and infinite sets

Relative notion of "size"

Thus, two finite/infinite sets have the same "size" iff there is a bijection between them.

Properties of finite and infinite sets

Similarities between finite and infinite sets

- $\exists$ **bij** from A to B and B to C, implies $\exists$ **bij** from A to C.
- $\exists$ **bij** from A to B, then $\exists$ **bij** from B to A.
- $\exists$ **surj** from A to B and $\exists$ **surj** B to A, implies $\exists$ **bij** from A to B.

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- If A, B are infinite, we can have $A \subsetneq B$, and a bijection from A to B, i.e., they have the same "cardinality".
- From any infinite set A, there is a surjection from A to $\mathbb{N}$.

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One direction is easy. ($\Longleftarrow$)

1. Show contrapositive! If A is finite, then there can't be a bijection from A to $A \cup \{b\}$.

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3. $\forall i \in \mathbb{N}$, $i \geq 1$, $A \setminus \{a_0, \ldots a_{i-1}\}$ is infinite, hence non-empty, so let $a_i \in A \setminus \{a_0, \ldots, a_{i-1}\}$.

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4. Collecting all such $a_i's$, we get a subset $A' = \{a_i \in A \mid i \in \mathbb{N}\} \subseteq A$. (Note it may be that $A \neq A'$).
5. Now, $\forall a \in A$ if $a \notin A'$, we define $f(a) = a$.

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6. f is a bijection! Prove surjection and injection...

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Some questions...

1. Is there a bijection between $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$?
2. Is there a bijection between $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$?
3. Is there a bijection from $\mathbb{Q}$ to $\mathbb{N}$?
4. Is there a bijection from the set of all subsets of $\mathbb{N}$ to $\mathbb{N}$?
5. Is there a bijection from $\mathbb{R}$ to $\mathbb{N}$?

Countable and countably infinite sets

Definition

- For a given set C, if there is a bijection from C to $\mathbb{N}$, then C is called countably infinite.
- A set is countable if it is finite or countably infinite.

Examples: even numbers, number of horses,...

By previous corollary ($\exists$ surj from any infinite set to $\mathbb{N}$)

Countably infinite sets are the "smallest" infinite sets.

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Countably infinite sets are the “smallest” infinite sets.

What are the other properties of countable sets?

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Are the following sets countable?

That is, is there a bijection from these sets to $\mathbb{N}$?

- the set of all integers $\mathbb{Z}$
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To show these it suffices to show that

- there is an injection from these sets to $\mathbb{N}$
- or there is a surjection from $\mathbb{N}$ (or any countable set) to these sets.

Unions of countable sets is countable

Let $A = \{a_0, \ldots, \}$ be a countably infinite set and B be a set. Then, is $A \cup B$ countable, under the following conditions?

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- Is this correct?

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Hint: Show that $f(a, b) = \begin{cases} a/b \text{ if } b \neq 0 \\ 0 \text{ if } b = 0 \end{cases}$, is a surjection. How does the result follow?

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