CS 105: Department Introductory Course on Discrete Structures

Instructor : S. Akshay

Aug 22, 2024

Lecture 11 – Basic structures: Relations

Logistics

Quiz 1

- VENUE: LH 101, 102, 301, 302
- Date and time: Aug 28th, 8.25am

Summary and moving on...

Week 01 and 02: Proofs and Reasoning

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- Theorems and Types of Proofs
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- Finite and infinite sets.
- Using functions to compare sets: focus on bijections.
- Countable, countably infinite and uncountable sets.
- Cantor's diagonalization argument (A new powerful proof technique!).

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Next: Basic Mathematical Structures – Relations

Relations

Definition: Function

Let A, B be two sets. A function f from A to B is a subset R of $A \times B$ such that

(i) $\forall a \in A$, $\exists b \in B$ such that $(a, b) \in R$, and

(ii) if $(a, b) \in R$ and $(a, c) \in R$, then $b = c$.

- Now, suppose A is the set of all Btech students and B is the set of all courses. Clearly, we can assign to each student the set of courses he/she is taking. Is this a function?

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- By removing the two extra assumptions in the defn, we get:

Definition: Relation

- A relation R from A to B is a subset of $A \times B$. If $(a, b) \in R$, we also write this as $a\ R\ b$.
- Thus, a relation is a way to relate the elements of two (not necessarily different) sets.

Examples and representations of relations

We write $R(A, B)$ for a relation from A to B and just $R(A)$ if $A = B$. Also if A is clear from context, we just write R.

Examples of relations

- All functions are relations.
- $R_1(\mathbb{Z}) = \{(a, b) \mid a, b \in \mathbb{Z}, a - b \text{ is even } \}$.
- $R_2(\mathbb{Z}) = \{(a, b) \mid a, b \in \mathbb{Z}, a \leq b\}$.
- Let S be a set, $R_3(\mathcal{P}(S)) = \{(A, B) \mid A, B \subseteq S, A \subseteq B\}$.
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Representations of a relation from A to B.

- As a set of ordered pairs of elements, i.e., subset of $A \times B$.
- As a directed graph.
- As a (database) table.

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  - Equivalence relations
  - Partial orders

Partitions of a set – grouping “like” elements

Examples

- Natural numbers are partitioned into even and odd.
- This class is partitioned into sets of students from same hostel.

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Can you think of two trivial partitions that any set must have?

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What properties does this relation have?

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Definition

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Thus, from any partition, we get an equivalence relation. Is the converse true?

Examples

- Reflexive: $\forall a \in S$, aRa.
- Symmetric: $\forall a, b \in S$, aRb implies bRa.
- Transitive: $\forall a, b, c \in S$, aRb, bRc implies aRc.
- Equivalence: Reflexive, Symmetric and Transitive.

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- Equivalence: Reflexive, Symmetric and Transitive.

Relation	Refl.	Sym.	Trans.	Equiv.
aR_4b if students a and b take same set of courses	✓	✓	✓	✓
aR_5b if student a takes course b				
$\{(a,b) \mid a,b \in \mathbb{Z}, (a-b) \mod 2 = 0\}$				
$\{(a,b) \mid a,b \in \mathbb{Z}, a \leq b\}$				
$\{(a,b) \mid a,b \in \mathbb{Z}, a < b\}$				
$\{(a,b) \mid a,b \in \mathbb{Z}, a \mid b\}$				
$\{(a,b) \mid a,b \in \mathbb{R}, \lvert a-b \rvert < 1\}$				
$\{((a,b),(c,d)) \mid (a,b),(c,d) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}), (ad = bc)\}$				

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Is the converse true? Can we generate a partition from every equivalence relation?