CS 105: Department Introductory Course on Discrete Structures

Instructor : S. Akshay

Aug 26, 2024

Lecture 12 – Basic structures: Equivalence relations

Logistics

Quiz 1

- Date and time: Aug 28th, Wednesday, 8.25am
- Syllabus: All material till last and including last lecture!
- VENUE:
  - LH 101, 102
  - PwD students: LT 105

Recap and moving on...

Week 01 and 02: Proofs and Reasoning

- Propositions, Predicates, Quantifiers
- Theorems and Types of Proofs
- Induction and variants

Week 03 and 04: Basic Mathematical Structures

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- Finite and infinite sets.
- Using functions to compare sets: focus on bijections.
- Countable, countably infinite and uncountable sets.
- Cantor's diagonalization argument (A new powerful proof technique!).

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Basic Mathematical Structures – Relations

Partitions of a set – grouping “like” elements

Definition

A partition of a set S is a set P of its subsets such that

- if $S' \in P$, then $S' \neq \emptyset$.
- $\bigcup_{S' \in P} S' = S$: its union covers entire set S.
- If $S_1, S_2 \in P$, then $S_1 \cap S_2 = \emptyset$: sets are disjoint.

Example: natural numbers partitioned into even and odd...

Theorem

Every partition of set S gives rise to a canonical equivalence relation R on S, namely,

- aRb if a and b belong to the same set in the partition of S.

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Is the converse true? Can we generate a partition from every equivalence relation?

Equivalence classes

Definition

- Let R be an equivalence relation on set S, and let $a \in S$.
- Then the equivalence class of a, denoted $[a]$, is the set of all elements related to it, i.e., $[a] = \{b \in S \mid (a, b) \in R\}$.

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Let R be an equivalence relation on S. Let $a, b \in S$. Then, the following statements are equivalent:

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Proof Sketch: (1) to (2) symm and trans, (2) to (3) refl, (3) to (1) symm and trans. (H.W.: Rewrite the proof formally.)

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Proof sketch of (1): Union, non-emptiness follows from reflexivity. The rest (pairwise disjointness) follows from the previous lemma.
(H.W.): Write the formal proofs of (1) and (2).

More “applications” of equivalence relations

Defining new objects using equivalence relations

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$R = \{((a,b),(c,d)) \mid (a,b),(c,d) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}), (ad = bc)\}.$

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- Then the equivalence classes of R define the rational numbers.
- e.g., $\left[\frac{1}{2}\right] = \left[\frac{2}{4}\right]$ are two names for the same rational number.
- Indeed, when we write $\frac{p}{q}$ we implicitly mean $\left[\frac{p}{q}\right]$.

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Can we define integers and real numbers starting from naturals by using equivalence classes?

Geometrical objects using equivalence relations

Cut-and-paste

Consider the relation $R([0,1]) = \{aRb \mid a, b \in [0,1]$, either $a = b$ or $a = 1, b = 0$, or $a = 0, b = 1\}$.

- Is R an equivalence relation? What does it define?

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- This is $[0,1]$ in which the end-points have been related to each other.
- So the equivalence classes form a "loop", since end-points are joined. If we imagine $[0,1]$ as a 1-length string, we have glued its ends!

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Can you build even more interesting “shapes”? Torus? Mobius strip?!