CS 105: Department Introductory Course on Discrete Structures

Instructor : S. Akshay

Aug 27, 2024

Lecture 13 – Basic structures: relations: posets

Logistics

Quiz 1

- Date and time: Aug 28th, Wednesday, 8.25am
- Syllabus: All material till last and including last lecture!
- VENUE:
  - LH 101, 102
  - PwD students: LT 105

Recap and moving on...

Week 01 and 02: Proofs and Reasoning

- Propositions, Predicates, Quantifiers
- Theorems and Types of Proofs
- Induction and variants

Week 03 and 04: Basic Mathematical Structures

- Finite and infinite sets.
- Using functions to compare sets: focus on bijections.
- Countable, countably infinite and uncountable sets.
- Cantor's diagonalization argument (A new powerful proof technique!).

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Basic Mathematical Structures – Relations

- Equivalence relations and partitions of a set

More “applications” of equivalence relations

Defining new objects using equivalence relations

Consider

$R = \{((a,b),(c,d)) \mid (a,b),(c,d) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}), (ad = bc)\}.$

More "applications" of equivalence relations

Defining new objects using equivalence relations

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- Then the equivalence classes of R define the rational numbers.
- e.g., $\left[\frac{1}{2}\right] = \left[\frac{2}{4}\right]$ are two names for the same rational number.
- Indeed, when we write $\frac{p}{q}$ we implicitly mean $\left[\frac{p}{q}\right]$.

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Can we define integers and real numbers starting from naturals by using equivalence classes?

Geometrical objects using equivalence relations

Cut-and-paste

Consider the relation $R([0,1]) = \{aRb \mid a, b \in [0,1]$, either $a = b$ or $a = 1, b = 0$, or $a = 0, b = 1\}$.

- Is R an equivalence relation? What does it define?

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- Is R an equivalence relation? What does it define?
- This is $[0,1]$ in which the end-points have been related to each other.
- So the equivalence classes form a "loop", since end-points are joined. If we imagine $[0,1]$ as a 1-length string, we have glued its ends!

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Can you build even more interesting "shapes"? Torus? Mobius strip?!

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Forming 2D objects

Consider a rectangular piece of the real plane, $[0,1] \times [0,1]$.

- Define $R_1([0,1] \times [0,1])$ by $(a,b)R_1(c,d)$ if
  - $(a,b) = (c,d)$ or
  - $b = d$, $a = 0$, $c = 1$ or
  - $b = d$, $c = 0$, $a = 1$.

  Is R_1 an equivalence relation? What do its equivalence classes define?

Geometrical objects using equivalence relations

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  - $b = d$, $c = 0$, $a = 1$.

  Is R_1 an equivalence relation? What do its equivalence classes define?
- Define $R_2([0, 1] \times [0, 1])$ by $(a, b)R_2(c, d)$ if
  - $(a, b) = (c, d)$ or
  - $a, b, c, d \in \{0, 1\}$.

  Is R_2 an equivalence relation? What does it define?

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Moving on another special relation: Partial Orders

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Anti-symmetric

A relation R on S is anti-symmetric if for all $a, b \in S$ aRb and bRa implies $a = b$.

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Examples:

- $R_1(\mathbb{Z}) = \{(a,b) \mid a,b \in \mathbb{Z}, a \leq b\}$.
- $R_2(\mathcal{P}(S)) = \{(A,B) \mid A,B \in \mathcal{P}(S), A \subseteq B\}$.

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Definition

A partial order is a relation which is reflexive, transitive and anti-symmetric.

Partial orders and equivalences relations

- Reflexive: $\forall a \in S$, aRa.
- Symmetric: $\forall a, b \in S$, aRb implies bRa.
- Anti-symmetric: $\forall a, b \in S$, aRb, bRa implies $a = b$.
- Transitive: $\forall a, b, c \in S$, aRb, bRc implies aRc.

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	Reflexive	Transitive	Symmetric	Anti-symmetric
Equivalence relation	✓	✓	✓	
Partial order	✓	✓		✓

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	Refl.	Anti-Sym	Trans.	PO
$\{(a, b) \mid a, b \in \mathbb{Z}, a \leq b\}$	✓	✓	✓	✓
$\{(A, B) \mid A, B \in \mathcal{P}(S), A \subseteq B\}$	✓	✓	✓	✓
$\{(a, b) \mid a, b \in \mathbb{Z}, a < b\}$				
$\{(a, b) \mid a, b \in \mathbb{Z}^+, a \mid b\}$				
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- A total order is a partial order $\preceq$ on S in which every pair of elements is comparable
  - i.e., $\forall a, b \in S$, either $a \preceq b$ or $b \preceq a$.

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$\{(A,B) \mid A,B \in \mathcal{P}(S), A \subseteq B\}$	✓	✓	✓	✓	×
$\{(a,b) \mid a,b \in \mathbb{Z}, a < b\}$					
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- Qn: Can a relation be symmetric and anti-symmetric?

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- A total order is a partial order $\preceq$ on S in which every pair of elements is comparable
- Qn: Can a relation be symmetric and anti-symmetric?
- Qn: Can a relation be neither symmetric nor anti-symmetric?

Partially ordered sets (Posets)

Definition

A set S together with a partial order $\preceq$ on S, is called a partially-ordered set or poset, denoted $(S, \preceq)$.

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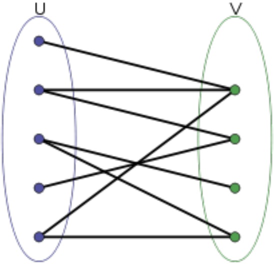
Examples

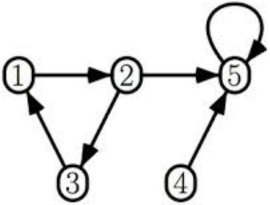
- $(\mathbb{Z}, \leq)$: integers with the usual less than or equal to relation.
- $(\mathcal{P}(S), \subseteq)$: powerset of any set with the subset relation.
- $(\mathbb{Z}^+, |\ )$: positive integers with divisibility relation.

Graphical representation of relations: posets

Recall: any relation on a set can be represented as a graph with

- nodes as elements of the set and
- directed edges between them indicating the ordered pairs that are related.





- Did these come from posets?
- Do graphs defined by posets have any "special" properties?