CS 105: Department Introductory Course on Discrete Structures

Instructor : S. Akshay

Aug 29, 2024

Lecture 14 – Basic structures: posets

Summary

Week 01 and 02: Proofs and Reasoning

- Propositions, Predicates, Quantifiers
- Theorems and Types of Proofs
- Induction and variants

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Week 03 and 04: Basic Structures – Sets and Functions

- Finite and infinite sets.
- Using functions to compare sets: focus on bijections.
- Countable, countably infinite and uncountable sets.
- Cantor's diagonalization argument (A new powerful proof technique!).

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Week 05: Basic Structures – Relations

- Equivalence relations and partitions of a set
- Partially ordered sets (posets)

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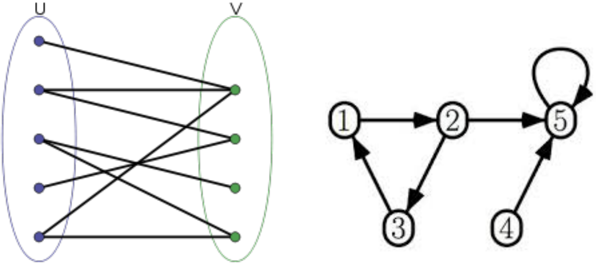
Examples

- $(\mathbb{Z}, \leq)$: integers with the usual less than or equal to relation.
- $(\mathcal{P}(S), \subseteq)$: powerset of any set with the subset relation.
- $(\mathbb{Z}^+, \mid\,)$: positive integers with divisibility relation.

Graphical representation of relations: posets

Recall: any relation on a set can be represented as a graph with

- nodes as elements of the set and
- directed edges between them indicating the ordered pairs that are related.



- Did these come from posets?
- Do graphs defined by posets have any "special" properties?

Graphical representation of relations: posets

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- How does the graph of $(\mathcal{P}(S), \subseteq)$ look like?

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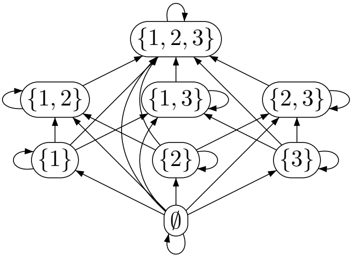

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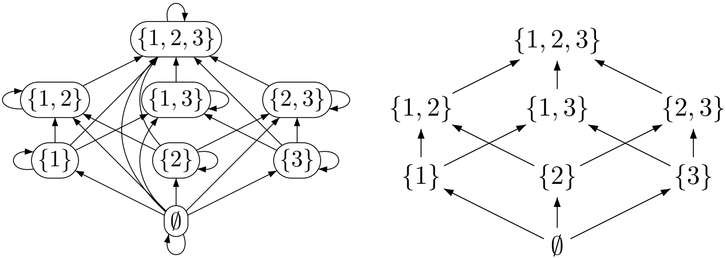

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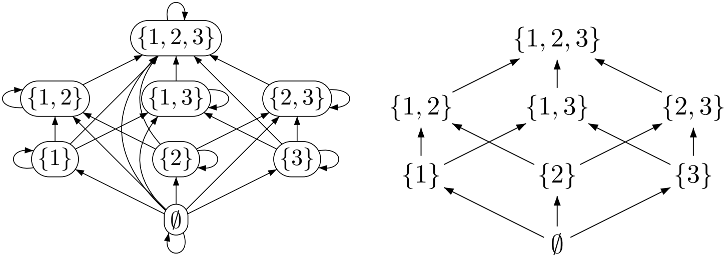

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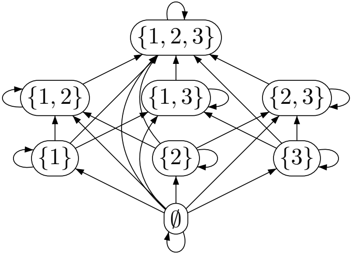

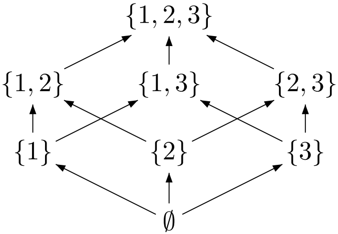

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- Starting from a node and following the directed edges (except self-loops), one can't come back to the same node.

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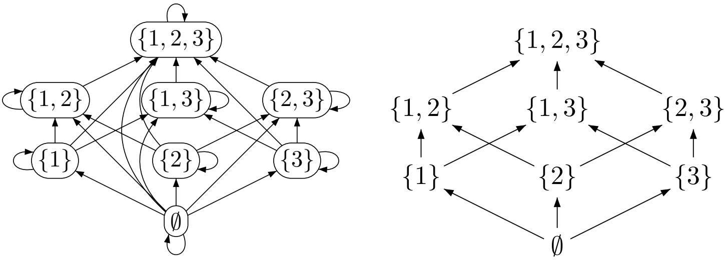

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- What is "special" about these graphs?
- Graphs of posets are "acyclic" (except for self-loops).
- Starting from a node and following the directed edges (except self-loops), one can't come back to the same node.
- Given the Hasse diagram of a poset, its reflexive transitive closure gives back the graph of the poset.

Chains and Anti-chains

Definition

Let $(S, \preceq)$ be a poset. A subset $B \subseteq S$ is called

- a chain if every pair of elements in B is related by $\preceq$.
- That is, $\forall a, b \in B$, we have $a \preceq b$ or $b \preceq a$ (or both).

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- an anti-chain if no two distinct elements of A are related to each other under $\preceq$.
- That is, $\forall a, b \in A, a \neq b$, we have neither $a \preceq b$ nor $b \preceq a$.

Chains and Anti-chains: examples

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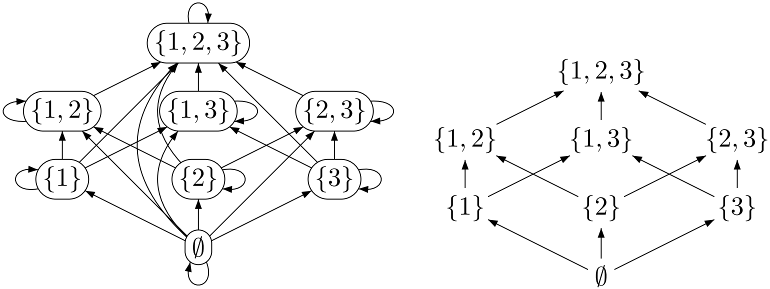

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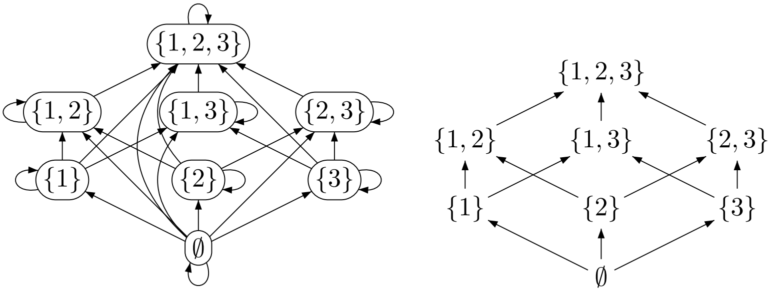

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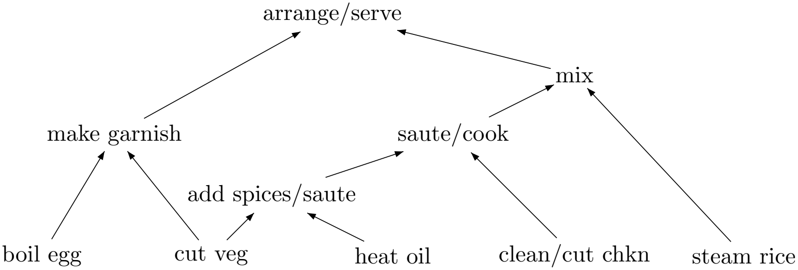
Examples and applications

A task scheduling example

Let us represent a recipe for making Chicken Biriyani as a poset!



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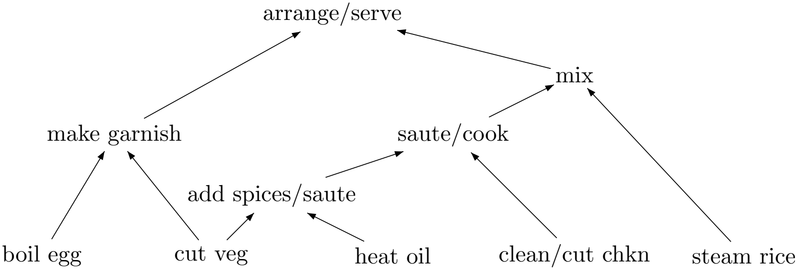

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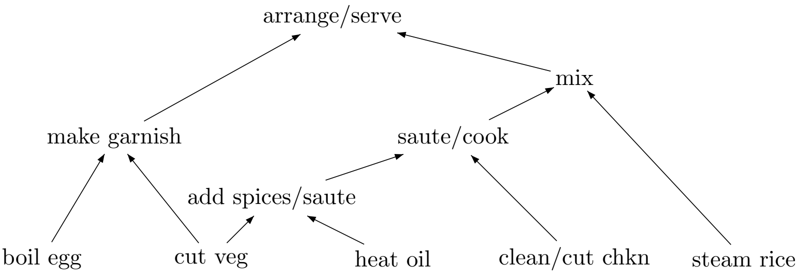

- Clearly, this shows the dependencies.
- But when you cook you need a total order, right?
- Further, this total order must be consistent with the po.
- This is called a linearization or a topological sorting.

Topological sorting

Definition

A topological sort or a linearization of a poset $(S, \preceq)$ is a poset $(S, \preceq_t)$ with a total order $\preceq_t$ such that $x \preceq y$ implies $x \preceq_t y$.

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- Then, construct the chain to complete the proof.

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Proof by induction?(H.W)

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Give an example on the poset seen earlier!