CS 105: DIC on Discrete Structures

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Sept 03, 2024

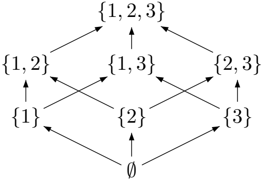
Lecture 16 – a little bit on lattices and on to Counting

Recap: Partial order relations

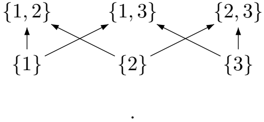
Last two classes we saw

- Partial orders: definition and examples
- Posets, chains and anti-chains
- Graphical representation as Directed Acyclic Graphs
- Topological sorting (application to task scheduling)
- Mirsky's theorem (application to parallel task scheduling)

Minimal and maximal elements



Poset $P_1 = (S_1, \subseteq)$
where $S_1 = \mathcal{P}(\{1,2,3\})$



Poset $P_2 = (S_2, \subseteq)$
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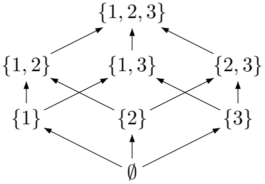

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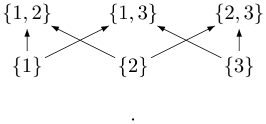

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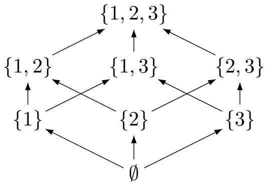

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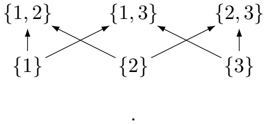

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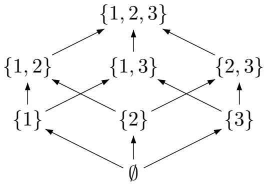

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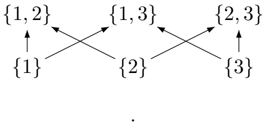

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Examples: $\emptyset$ is a minimal and the minimum element in P_1, $\{1\}, \{2\}, \{3\}$ are all minimal elements in P_2, but P_2 does not have any minimum element.

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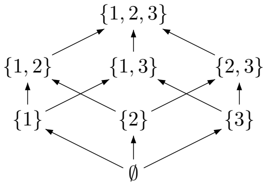

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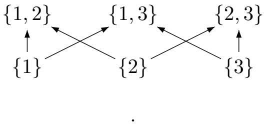

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Exercise: What are the maximal/maximum elements in P_1, P_2?

Least upper bounds and greatest lower bounds

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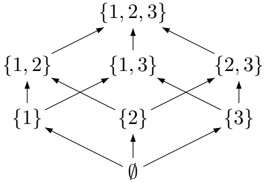

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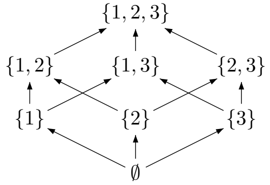

Poset $P_1 = (S_1, \subseteq)$

- Let $A = \{\{1\}, \{2\}\}$. Then $\{1, 2\}, \{1, 2, 3\}$ are upper bounds of A in P_1 and $\{1, 2\}$ is the lub of A.

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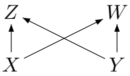

Poset $P_3 = (S_3, \preceq)$

- Consider $P_3 = (S_3, \preceq)$ where $S_3 = \{X, Y, Z, W\}$ and the $\preceq$ is as given by the arrows. Let $B = \{X, Y\}$. Then Z, W are both upper bounds of B in P_3, but B has no lub in P_3.

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Some Obervations (Exercise: Prove it!)

- The lub/glb of a subset A in S, if it exists, is unique.
- If the lub/glb of $A \subseteq S$ belongs to A, then it is the greatest/least element of A.

Lattices

Definition

- A lattice is a poset in which every pair of elements has both a lub and a glb (in the set), i.e., $\forall x, y \in S$, there exists $l, u \in S$ such that l is the glb and u is the lub of $\{x, y\}$.

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Applications of Lattices

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- Finite lattices have a strong link with Boolean Algebra
- Several other applications in many domains of mathematics and CS, including formal semantics of programming languages, program verification.

Summary till now

Course Outline

1. Proofs and reasoning
2. Basic discrete structures
3. Counting and combinatorics
4. Introduction to graph theory

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    - (finite and infinite) sets
    - Functions: injections, surjections, bijections
    - Relations: equivalence relations, partial orders, lattices
    - Some applications
        - Functions: To compare infinite sets
        - Using diagonalization to prove impossibility results.
        - Equivalences: Defining "like" partitions.
        - Posets: Topological sort, (parallel) task scheduling, lattices
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Pop Quiz

Pop Quiz

Fill the feedback form at

`https://forms.gle/MpASeXVG9YqChonZ6`

Next chapter: Counting and Combinatorics

Topics to be covered

- Basics of counting
- Subsets, partitions, Permutations and combinations
- Pigeonhole Principle and its extensions
- Recurrence relations and generating functions

Introduction to combinatorics

Does it really need an introduction

Introduction to combinatorics

Does it really need an introduction

- Enumerative combinatorics: counting combinatorial/discrete objects e.g., sets, numbers, structures...
- Existential combinatorics: show that there exist some combinatorial "configurations".
- Constructive combinatorics: construct interesting configurations...