CS 105: DIC on Discrete Structures

Instructor : S. Akshay

Sept 05, 2024

Lecture 17 – Counting

Summary till now

Course Outline

1. Proofs and reasoning
2. Basic discrete structures
3. Counting and combinatorics
4. Introduction to graph theory

Summary till now

Course Outline

1. Proofs and reasoning
   - Propositions, predicates
   - Proofs and proof techniques: contradiction, contrapositive, (strong) induction, well-ordering principle, diagonalization.
2. Basic discrete structures
3. Counting and combinatorics
4. Introduction to graph theory

Summary till now

Course Outline

1. Proofs and reasoning
   - Propositions, predicates
   - Proofs and proof techniques: contradiction, contrapositive, (strong) induction, well-ordering principle, diagonalization.
2. Basic discrete structures
   - (finite and infinite) sets
   - Functions: injections, surjections, bijections
   - Relations: equivalence relations, partial orders, lattices
   - Some applications
     - Functions: To compare infinite sets
     - Using diagonalization to prove impossibility results.
     - Equivalences: Defining "like" partitions.
     - Posets: Topological sort, (parallel) task scheduling, lattices
3. Counting and combinatorics
4. Introduction to graph theory

Summary till now

Course Outline

1. Proofs and reasoning
   - Propositions, predicates
   - Proofs and proof techniques: contradiction, contrapositive, (strong) induction, well-ordering principle, diagonalization.
2. Basic discrete structures
   - (finite and infinite) sets
   - Functions: injections, surjections, bijections
   - Relations: equivalence relations, partial orders, lattices
   - Some applications
3. Counting and combinatorics
4. Introduction to graph theory

Next chapter: Counting and Combinatorics

Topics to be covered

- Basics of counting
- Subsets, partitions, Permutations and combinations
- Recurrence relations and generating functions
- Pigeonhole Principle and its extensions

Introduction to combinatorics

Various aspects of combinatorics

Introduction to combinatorics

Various aspects of combinatorics

- Enumerative combinatorics: counting combinatorial/discrete objects e.g., sets, numbers, structures...
- Existential combinatorics: show that there exist some combinatorial "configurations".
- Constructive combinatorics: construct interesting configurations...

Simple examples and principles...

- How many reflexive relations are there on a set A of size n?

Simple examples and principles...

- How many reflexive relations are there on a set A of size n?
  - Reflexive relations are ordered pairs of which there are n^2.
  - Of these, all n pairs of (a, a) have be present.
  - Of the remaining, we can choose any of them to be in or out.
  - there are $n^2 - n$ of them, so 2^{n^2-n} of them.
  - We used the so-called "product principle"...

Simple examples and principles...

- How many reflexive relations are there on a set A of size n?
  - Reflexive relations are ordered pairs of which there are n^2.
  - Of these, all n pairs of (a, a) have be present.
  - Of the remaining, we can choose any of them to be in or out.
  - there are $n^2 - n$ of them, so 2^{n^2-n} of them.
  - We used the so-called "product principle"...

The product principle

If there are n_1 ways of doing something and n_2 ways of doing another thing, then there are $n_1 \cdot n_2$ ways of performing both actions.

Simple examples and principles...

- How many reflexive relations are there on a set A of size n?

Simple examples and principles...

- How many reflexive relations are there on a set A of size n?

- How many functions are there from a set of size n to itself?

Simple examples and principles...

- How many reflexive relations are there on a set A of size n?

- How many functions are there from a set of size n to itself?

- How many subsets does a set A of n elements have?

Simple examples and principles...

- How many reflexive relations are there on a set A of size n?

- How many functions are there from a set of size n to itself?

- How many subsets does a set A of n elements have?
  - Product principle: two choices for each element, hence $2 \cdot 2 \cdots 2 \cdot 2$ (n-times).
  - Bijection: between $\mathcal{P}(A)$ and n-length sequences over $\{0, 1\}$ (characteristic vector).
  - Induction: Since we already know the answer!
  - Recurrence: $F(n) = 2 \cdot F(n-1), F(0) = 1$. solve it?
  - Sum principle: Subsets of size 0 + subsets of size 1 + ... + subsets of size n = Total number of subsets.

Simple examples and principles...

- How many reflexive relations are there on a set A of size n?
- How many functions are there from a set of size n to itself?
- How many subsets does a set A of n elements have?
  - Product principle: two choices for each element, hence $2 \cdot 2 \cdots 2 \cdot 2$ (n-times).
  - Bijection: between $\mathcal{P}(A)$ and n-length sequences over $\{0, 1\}$ (characteristic vector).
  - Induction: Since we already know the answer!
  - Recurrence: $F(n) = 2 \cdot F(n-1), F(0) = 1$. solve it?
  - Sum principle: Subsets of size 0 + subsets of size 1 + $\ldots$ + subsets of size n = Total number of subsets.

Sum Principle

If something can be done in n_1 or n_2 ways such that none of the n_1 ways is the same as any of the n_2 ways, then the total number of ways to do this is $n_1 + n_2$.

Simple examples and principles...

- How many reflexive relations are there on a set A of size n?

- How many functions are there from a set of size n to itself?

- How many subsets does a set A of n elements have?
  - Product principle: two choices for each element, hence $2 \cdot 2 \cdots 2 \cdot 2$ (n-times).
  - Bijection: between $\mathcal{P}(A)$ and n-length sequences over $\{0, 1\}$ (characteristic vector).
  - Induction: Since we already know the answer!
  - Recurrence: $F(n) = 2 \cdot F(n-1), F(0) = 1$. solve it?
  - Sum principle: Subsets of size 0 + subsets of size 1 + ... + subsets of size n = Total number of subsets.

- But, how many subsets of size k does a set of n elements have? This number, denoted $\binom{n}{k}$, is called a binomial coefficient.

Simple examples and principles...

- How many reflexive relations are there on a set A of size n?

- How many functions are there from a set of size n to itself?

- How many subsets does a set A of n elements have?
  - Product principle: two choices for each element, hence $2 \cdot 2 \cdots 2 \cdot 2$ (n-times).
  - Bijection: between $\mathcal{P}(A)$ and n-length sequences over $\{0, 1\}$ (characteristic vector).
  - Induction: Since we already know the answer!
  - Recurrence: $F(n) = 2 \cdot F(n-1), F(0) = 1$. solve it?
  - Sum principle: Subsets of size 0 + subsets of size 1 + ... + subsets of size n = Total number of subsets.

- But, how many subsets of size k does a set of n elements have? This number, denoted $\binom{n}{k}$, is called a binomial coefficient.
- We all know(?) that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Prove it!

Permutations and combinations

Binomial Coefficients. Let n, k be integers s.t., $n \geq k \geq 0$.

How many subsets of size k does a set of n elements have? This number, denoted $\binom{n}{k}$, is called a binomial coefficient.

One proof of $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is as follows:

Permutations and combinations

Binomial Coefficients. Let n, k be integers s.t., $n \geq k \geq 0$.

How many subsets of size k does a set of n elements have? This number, denoted $\binom{n}{k}$, is called a binomial coefficient.

One proof of $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is as follows: Let us count the number of "ordered" subsets of size k.

Permutations and combinations

Binomial Coefficients. Let n, k be integers s.t., $n \geq k \geq 0$.

How many subsets of size k does a set of n elements have? This number, denoted $\binom{n}{k}$, is called a binomial coefficient.

One proof of $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is as follows: Let us count the number of "ordered" subsets of size k.

- No. of ordered subsets of size k= $n \cdot (n-1) \cdots (n-k+1)$.

Permutations and combinations

Binomial Coefficients. Let n, k be integers s.t., $n \geq k \geq 0$.

How many subsets of size k does a set of n elements have? This number, denoted $\binom{n}{k}$, is called a binomial coefficient.

One proof of $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is as follows: Let us count the number of "ordered" subsets of size k.

- No. of ordered subsets of size k= $n \cdot (n-1) \cdots (n-k+1)$.
- No. of ordered subsets of size k = (no. of unordered subsets)×(no. of ways to order them)=$\binom{n}{k} \times k!$.

Permutations and combinations

Binomial Coefficients. Let n, k be integers s.t., $n \geq k \geq 0$.

How many subsets of size k does a set of n elements have? This number, denoted $\binom{n}{k}$, is called a binomial coefficient.

One proof of $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is as follows: Let us count the number of "ordered" subsets of size k.

- No. of ordered subsets of size $k = n \cdot (n-1) \cdots (n-k+1)$.
- No. of ordered subsets of size k = (no. of unordered subsets)×(no. of ways to order them)$= \binom{n}{k} \times k!$.
- Equate them! Principle of double counting.
  - if you can't count something, count something else and count it twice over!

Permutations and combinations

Binomial Coefficients. Let n, k be integers s.t., $n \geq k \geq 0$.

How many subsets of size k does a set of n elements have? This number, denoted $\binom{n}{k}$, is called a binomial coefficient.

One proof of $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is as follows: Let us count the number of "ordered" subsets of size k.

- No. of ordered subsets of size $k = n \cdot (n-1) \cdots (n-k+1)$.
- No. of ordered subsets of size k = (no. of unordered subsets)×(no. of ways to order them)=$\binom{n}{k} \times k!$.
- Equate them! Principle of double counting.

Permutations and combinations

- No. of k-size subsets of set of size n = No. of k-combinations of a set of n (distinct) elements = $\binom{n}{k}$.
- No. of k-size ordered subsets of set of size n = No. of k-permutations of a set of n (distinct) elements.

Simple examples to illustrate "double counting"

Prove the following identities (by only using double counting!). For all $n, k \in \mathbb{N}$ with $0 \leq k \leq n$,

1. $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n.$
2. $\displaystyle\binom{n}{k} = \binom{n}{n-k}.$
3. $\displaystyle k\binom{n}{k} = n\binom{n-1}{k-1}$
4. $\displaystyle\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

Simple examples to illustrate "double counting"

Prove the following identities (by only using double counting!). For all $n, k \in \mathbb{N}$ with $0 \leq k \leq n$,

1. $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$.
2. $\displaystyle\binom{n}{k} = \binom{n}{n-k}$.
3. $\displaystyle k\binom{n}{k} = n\binom{n-1}{k-1}$
4. $\displaystyle\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

The latter two are in fact recursive definitions for $\binom{n}{k}$. What are the boundary conditions?

A more interesting example with double counting

Handshake Lemma

At a meeting with n people, the number of people who shake hands an odd number of times is even.

What will you count here?

A more interesting example with double counting

Handshake Lemma

At a meeting with n people, the number of people who shake hands an odd number of times is even.

Proof in nine steps:

1. Define a relation R: iRj if i and j shook hands.
2. Is this relation symmetric (trans/refl.)? Draw its graph.

A more interesting example with double counting

Handshake Lemma

At a meeting with n people, the number of people who shake hands an odd number of times is even.

Proof in nine steps:

1. Define a relation R: iRj if i and j shook hands.
2. Is this relation symmetric (trans/refl.)? Draw its graph.
3. Let m_i be the number of times i shakes hands.

A more interesting example with double counting

Handshake Lemma

At a meeting with n people, the number of people who shake hands an odd number of times is even.

Proof in nine steps:

1. Define a relation R: iRj if i and j shook hands.
2. Is this relation symmetric (trans/refl.)? Draw its graph.
3. Let m_i be the number of times i shakes hands. i.e., m_i is the number of directed edges going out from i.

A more interesting example with double counting

Handshake Lemma

At a meeting with n people, the number of people who shake hands an odd number of times is even.

Proof in nine steps:

1. Define a relation R: iRj if i and j shook hands.
2. Is this relation symmetric (trans/refl.)? Draw its graph.
3. Let m_i be the number of times i shakes hands. i.e., m_i is the number of directed edges going out from i.
4. Therefore, Total no. of directed edges $= \sum_i m_i$.

A more interesting example with double counting

Handshake Lemma

At a meeting with n people, the number of people who shake hands an odd number of times is even.

Proof in nine steps:

1. Define a relation R: iRj if i and j shook hands.
2. Is this relation symmetric (trans/refl.)? Draw its graph.
3. Let m_i be the number of times i shakes hands. i.e., m_i is the number of directed edges going out from i.
4. Therefore, Total no. of directed edges $= \sum_i m_i$.
5. But now, let X be the total number of handshakes. Clearly this is an integer.

A more interesting example with double counting

Handshake Lemma

At a meeting with n people, the number of people who shake hands an odd number of times is even.

Proof in nine steps:

1. Define a relation R: iRj if i and j shook hands.
2. Is this relation symmetric (trans/refl.)? Draw its graph.
3. Let m_i be the number of times i shakes hands. i.e., m_i is the number of directed edges going out from i.
4. Therefore, Total no. of directed edges $= \sum_i m_i$.
5. But now, let X be the total number of handshakes. Clearly this is an integer. Total no. of directed edges $= 2 \cdot X$.

A more interesting example with double counting

Handshake Lemma

At a meeting with n people, the number of people who shake hands an odd number of times is even.

Proof in nine steps:

1. Define a relation R: iRj if i and j shook hands.
2. Is this relation symmetric (trans/refl.)? Draw its graph.
3. Let m_i be the number of times i shakes hands. i.e., m_i is the number of directed edges going out from i.
4. Therefore, Total no. of directed edges $= \sum_i m_i$.
5. But now, let X be the total number of handshakes. Clearly this is an integer. Total no. of directed edges $= 2 \cdot X$.
6. i.e., $2 \cdot X = \sum_{i=1}^{n} m_i$

A more interesting example with double counting

Handshake Lemma

At a meeting with n people, the number of people who shake hands an odd number of times is even.

Proof in nine steps:

1. Define a relation R: iRj if i and j shook hands.
2. Is this relation symmetric (trans/refl.)? Draw its graph.
3. Let m_i be the number of times i shakes hands. i.e., m_i is the number of directed edges going out from i.
4. Therefore, Total no. of directed edges $= \sum_i m_i$.
5. But now, let X be the total number of handshakes. Clearly this is an integer. Total no. of directed edges $= 2 \cdot X$.
6. i.e., $2 \cdot X = \sum_{i=1}^{n} m_i = \sum_{\{i \mid m_i \text{ is odd}\}} m_i + \sum_{\{i \mid m_i \text{ is even}\}} m_i$

A more interesting example with double counting

Handshake Lemma

At a meeting with n people, the number of people who shake hands an odd number of times is even.

Proof in nine steps:

1. Define a relation R: iRj if i and j shook hands.
2. Is this relation symmetric (trans/refl.)? Draw its graph.
3. Let m_i be the number of times i shakes hands. i.e., m_i is the number of directed edges going out from i.
4. Therefore, Total no. of directed edges $= \sum_i m_i$.
5. But now, let X be the total number of handshakes. Clearly this is an integer. Total no. of directed edges $= 2 \cdot X$.
6. i.e., $2 \cdot X = \sum_{i=1}^{n} m_i = \sum_{\{i \mid m_i \text{ is odd}\}} m_i + \sum_{\{i \mid m_i \text{ is even}\}} m_i$
7. $\sum_{\{i \mid m_i \text{ is even}\}} m_i$ is even, as sum of even numbers is even.

A more interesting example with double counting

Handshake Lemma

At a meeting with n people, the number of people who shake hands an odd number of times is even.

Proof in nine steps:

1. Define a relation R: iRj if i and j shook hands.
2. Is this relation symmetric (trans/refl.)? Draw its graph.
3. Let m_i be the number of times i shakes hands. i.e., m_i is the number of directed edges going out from i.
4. Therefore, Total no. of directed edges $= \sum_i m_i$.
5. But now, let X be the total number of handshakes. Clearly this is an integer. Total no. of directed edges $= 2 \cdot X$.
6. i.e., $2 \cdot X = \sum_{i=1}^{n} m_i = \sum_{\{i \mid m_i \text{ is odd}\}} m_i + \sum_{\{i \mid m_i \text{ is even}\}} m_i$
7. $\sum_{\{i \mid m_i \text{ is even}\}} m_i$ is even, as sum of even numbers is even.
8. So $\sum_{\{i \mid m_i \text{ is odd}\}} m_i$ must also be even, but each of these m_i is odd, so $|\{i \mid m_i \text{ is odd}\}|$ has to be even.
9. Thus number of i such that m_i is odd is even! $\square$