CS 105: DIC on Discrete Structures

Instructor : S. Akshay

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Lecture 18 – Counting and Combinatorics

Next Chapter: Counting and Combinatorics

Topics to be covered

- Basics of counting
  - Product principle
  - Sum principle
  - Bijection principle
  - Double counting

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  1. Binomial coefficients and Binomial theorem
  2. Pascal's triangle
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- Recurrence relations and generating functions
- Pigeonhole Principle and its extensions

Recall: interesting example with double counting

Handshake Lemma

At a meeting with n people, the number of people who shake hands an odd number of times is even.

What will you count here?

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   Is this relation symmetric (trans/refl.)? Draw its graph.

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6. $\sum_{\{i \mid m_i \text{ is even}\}} m_i$ is even, as sum of even numbers is even.
   So $\sum_{\{i \mid m_i \text{ is odd}\}} m_i$ must also be even, but each of these m_i is odd, so $|\{i \mid m_i \text{ is odd}\}|$ has to be even.

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7. Thus number of i such that m_i is odd is even! $\square$

Binomial theorem

Recall: $\sum_{k=0}^{n} \binom{n}{k} = 2^n.$

We generalize this...

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Binomial Theorem

Let x, y be variables and $n \in \mathbb{Z}^{\geq 0}$. Then,

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$$(x+y)^1 = x + y$$

$$(x+y)^2 = (x+y)(x+y) = x^2 + 2xy + y^2$$

$$(x+y)^3 = (x+y)(x+y)^2 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = (x+y)(x+y)^3 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

...

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(H.W-1) Prove this by induction.

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Proof (Combinatorial):

1. Consider any term $x^i y^j$, where $i + j = n$.
2. To get $x^i y^j$ term in

$$(x+y)(x+y)\cdots(x+y) \qquad \text{(n times)}$$

   we need to pick j $y's$ from n sums and remaining x's.

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3. Thus, the coefficient of this term = number of ways to get this term = number of ways to pick j y's from n elts = $\binom{n}{j}$.

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Corollaries

1. $\binom{n}{j} = \binom{n}{n-j}$,
2. $\displaystyle\sum_{j=0}^{n} \binom{n}{j} 2^j = 3^n$.
3. No. of subsets of n-element set having even cardinality = No. of subsets of n-element set having odd cardinality (=?)

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Corollaries Ex: Using Binomial Thm show the following

1. $\binom{n}{j} = \binom{n}{n-j}$,
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Pascal's Triangle

A recursive way to compute binomial coefficients

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

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$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

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1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Fun with Pascal's triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 200 252 200 120 45 10 1

Some simple observations. Recall: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

1. Row i adds up to 2^i, Row $i+1$ adds up to twice of row i.
2. Sequence of numbers, squares, cubes?

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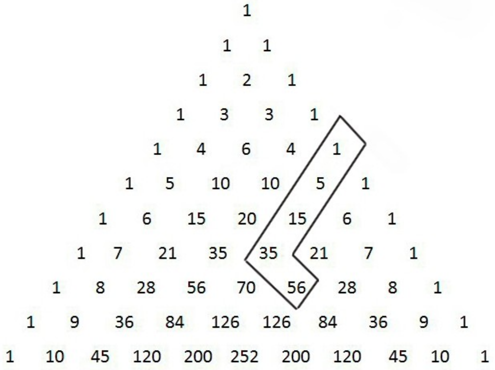

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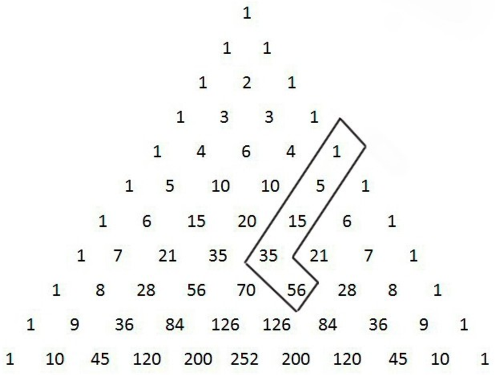

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1. Row i adds up to 2^i, Row $i+1$ adds up to twice of row i.
2. Sequence of numbers, squares, cubes?
3. Hockey stick patterns: (H.W-2)
   $\binom{n+1}{m} = \binom{n}{m} + \binom{n-1}{m-1} \ldots + \binom{n-m}{0}$

Fun with Pascal's triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

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