CS 105: DIC on Discrete Structures

Instructor : S. Akshay

Sept 09, 2024

Lecture 19 – Counting and Combinatorics

Counting and Combinatorics

Topics to be covered

- Basics of counting
  - Product principle
  - Sum principle
  - Bijection principle
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  1. Binomial coefficients and Binomial theorem
  2. Pascal's triangle
  3. Permutations and combinations with repetitions

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  2. Pascal's triangle
  3. Permutations and combinations with repetitions
- Recurrence relations and generating functions
- Pigeonhole Principle and its extensions

Fun with Pascal's triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 200 252 200 120 45 10 1

Some simple observations. Recall: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

1. Row i adds up to 2^i, Row $i+1$ adds up to twice of row i.
2. Sequence of numbers, squares, cubes?

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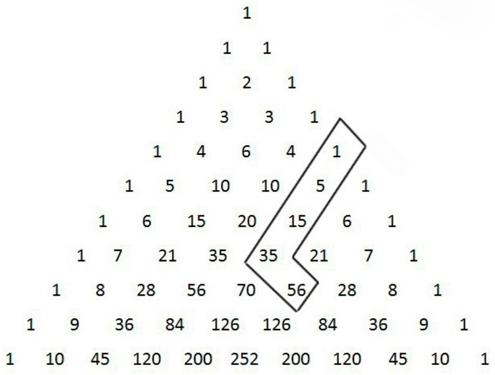

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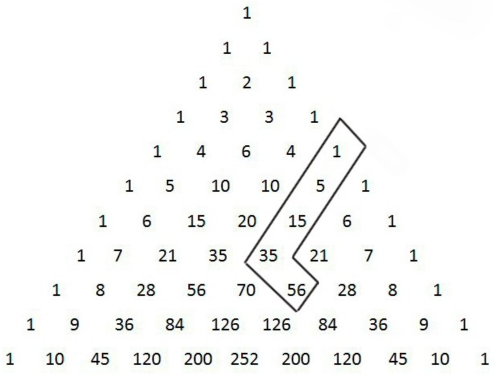

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2. Sequence of numbers, squares, cubes?
3. Hockey stick patterns: (H.W-2)
   $\binom{n+1}{m} = \binom{n}{m} + \binom{n-1}{m-1} \ldots + \binom{n-m}{0}$

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            1   4   6   4   1
          1   5  10  10   5   1
        1   6  15  20  15   6   1
      1   7  21  35  35  21   7   1
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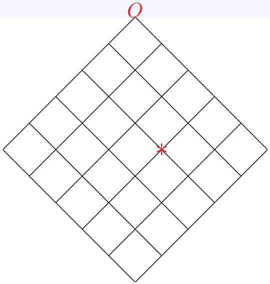
- Interesting Ex.: Count no. of odd numbers in each row...

An application to path counting

Map problems

From the top corner, how many shortest routes lead to a particular junction?



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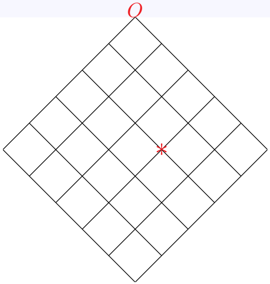

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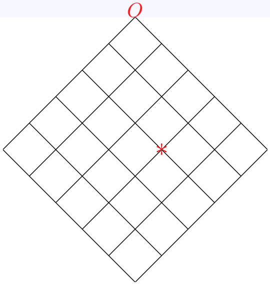

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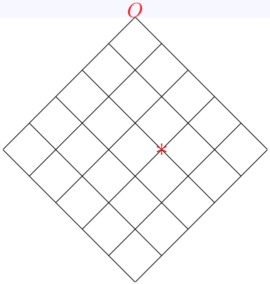

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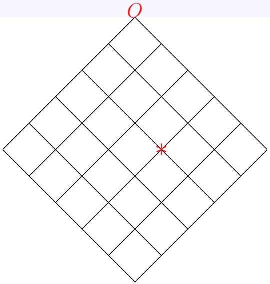

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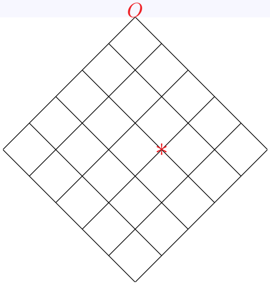

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H.W-3: Prove/verify this formally.

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- Ex: How many solutions does the equation $x_1 + x_2 + x_3 + x_4 = 17$ have such that $x_1, x_2, x_3, x_4 \in \mathbb{Z}^{\geq 0}$?

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Theorem (Stirling's Approximation)

$$e\left(\frac{n}{e}\right)^n \leq n! \leq ne\left(\frac{n}{e}\right)^n$$

where e is the base of natural logarithms, $\log(e) = e^{\log(e)} = 1$.